ЗАЩИТА ОТ ПЫЛИ

SYMBOLIC RELATIONS AMONG CLASSICAL POLYNOMIALS

E. D. RAINVILLE, University of Michigan

1. Introduction. We shall obtain several algebraic relations involving the polynomials of Legendre, Laguerre, and Hermite. Authors are not uniform in their notations for these polynomials. The notation used here is that implied by the following generating function definitions: For Legendre polynomials,

$$(1) \qquad (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n;$$

For Hermite polynomials,

$$(2) \qquad e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!};$$

For Laguerre polynomials,

$$(3) \qquad e^t J_0(2\sqrt{xt}) = \sum_{n=0}^{\infty} L_n(x)\frac{t^n}{n!},$$

where $J_0(x)$ is the Bessel function of the first kind and of index zero.

Explicit expressions for these polynomials are included for convenience. They are as follows:

$$(4) \qquad P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!x^{n-2k}}{2^n k!(n-k)!(n-2k)!},$$

where [ ] is the greatest integer symbol,

$$(5) \qquad H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!},$$

$$(6) \qquad L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n!x^k}{(k!)^2(n-k)!}.$$

The relations to be obtained here are particularly attractive in symbolic form. Whenever $\doteq$ is used to replace $=$, it is to be understood that exponents shall be lowered to subscripts on any symbol, such as P, H, L, which is undefined here except with subscripts. For example, since

$$L_2(x) = 1 - 2x + \tfrac{1}{2}x^2,$$

then

$$L_2(P(x)) \doteq P_0(x) - 2P_1(x) + \tfrac{1}{2}P_2(x).$$

In the same way

$$\{H(x) - P(x)\}^2 \doteq H_2(x)P_0(x) - 2H_1(x)P_1(x) + H_0(x)P_2(x).$$

2. Examples of known relations. Several of the known [1] algebraic relations involving the above polynomials may be expressed neatly in symbolic form. A few of them are:

$$H_n(x+y) \doteq \{H(x)+2y\}^n, \tag{7}$$

$$\{H(x)+H(y)\}^n \doteq 2^{n/2}H_n(2^{-1/2}(x+y)), \tag{8}$$

$$\{1-L(x)\}^n \doteq x^n/n!, \tag{9}$$

$$L_n(xy) \doteq \{1-y+yL(x)\}^n, \tag{10}$$

$$P_n(x) \doteq \{2x-P(x)\}^n, \tag{11}$$

$$P_n(1-2x^2) \doteq \{1-2xP(x)\}^n, \tag{12}$$

$$P_n(x) \doteq n!\left\{L\left(\frac{1-x}{2}\right)-L\left(\frac{1+x}{2}\right)\right\}^n. \tag{13}$$

3. Methods and results. We use three methods to obtain additional formulas. One method is the combination in an appropriate manner of equation (7) with itself or with other known relations. The second method is the application of a linear differential operator to a Hermite polynomial, followed by two distinct evaluations of the result. The third method is the manipulation of generating functions.

Each of the three methods will be exhibited in a representative derivation. For most of the results the proof will be omitted. In its place we shall give sufficient hints to enable one to supply the proof by one or more of the methods of this paper.

It is convenient to group our results here. The relations involving both the Hermite and the Legendre polynomials are:

$$H_n(P(x)) \doteq \{H(2x)-2P(x)\}^n, \tag{14}$$

$$H_n(\tfrac{1}{2}P(x)) \doteq \{H(x)-P(x)\}^n, \tag{15}$$

$$\{H(x)-2P(x)\}^{2n+1} \doteq 0, \tag{16}$$

$$\{H(x)-P(2x)\}^{2n+1} \doteq 0. \tag{17}$$

A relation involving the Legendre and the Laguerre polynomials is

$$2^nL_n(P(x)) \doteq \{L(x-1)+L(x+1)\}^n. \tag{18}$$

A relation involving the Hermite and the Laguerre polynomials is

$$H_n\left((1+x)L\left(\frac{x}{1+x}\right)\right) \doteq \{H(x)+2L(x)\}^n. \tag{19}$$

Relations in which all three, the Hermite, Legendre, and Laguerre polynomials enter are

$$n!\{H(x) - 2P(x)\}^{2n} \doteq (-1)^n(2n)!L_n(x^2 - 1), \tag{20}$$

$$n!\{H(x) - P(2x)\}^{2n} \doteq (-1)^n(2n!)L_n(x^2 - \tfrac{1}{4}). \tag{21}$$

Relations involving only the Hermite polynomials are:

$$H_n(\tfrac{1}{2}H(x + y)) \doteq \{H(x) + H(y)\}^n, \tag{22}$$

$$H_n(\tfrac{1}{2}H(x)) \doteq 2^{n/2}H_n(2^{-1/2}x), \tag{23}$$

$$H_n\left(\frac{w}{2}H(x)\right) \doteq w^nH_n\left(\frac{H(wx)}{2w}\right), \tag{24}$$

$$y^nH_n(x/y) \doteq (y^2 - 1)^{n/2}H_n\left(\frac{H(x)}{2\sqrt{y^2 - 1}}\right), \tag{25}$$

$$H_n(H(x)) \doteq 5^{n/2}H_n(2x/\sqrt{5}), \tag{25a}$$

$$H_n(xy) \doteq \{H(x) + 2x(y - 1)\}^n. \tag{26}$$

The Hermite polynomials and polynomials $g_n(x)$ studied earlier [2] are related by

$$H_n\left((1 + x)g\left(\frac{x}{1 + x}\right)\right) \doteq \{H(x) + 2g(x)\}^n. \tag{27}$$

A relation involving the Hermite polynomial and the hypergeometric polynomial $\phi_n(\alpha, \gamma, x) = F(\alpha, -n; \gamma; -x)$ of [2] is

$$H_n(\phi(\alpha, \gamma, x)) \doteq \{H(x) + 2\phi(\gamma - \alpha, \gamma, -x)\}^n. \tag{28}$$

It should be kept in mind that these relations are identities involving finite series in which the pertinent polynomials enter. The expanded form of equations (14) and (15), for example, is

$$\sum_{k=0}^{[n/2]} \frac{(-1)^k n!2^{n-2k}P_{n-2k}(x)}{k!(n - 2k)!} = \sum_{k=0}^{n} (-1)^k C_{n,k}2^k P_k(x)H_{n-k}(2x), \tag{14}$$

and

$$\sum_{k=0}^{[n/2]} \frac{(-1)^k n!P_{n-2k}(x)}{k!(n - 2k)!} = \sum_{k=0}^{n} (-1)^k C_{n,k}P_k(x)H_{n-k}(x), \tag{15}$$

in which $C_{n,k}$ is the binomial coefficient.

4. Derivation of the relation (14). Since equation (7) is an identity in y it follows for any set of polynomials $T_n(x)$ we have

$$H_n(x + T(x)) \doteq \{H(x) + 2T(x)\}^n. \tag{29}$$

Therefore

$$\{H(x) - 2P(x/2)\}^n \doteq H_n(x - P(x/2)),$$

in the right member of which various powers of $x-P(x/2)$ occur. Equation (11) with x replaced by $x/2$ shows that

$$\{x - P(x/2)\}^n \doteq P_n(x/2).$$

Hence

$$\{H(x) - 2P(x/2)\}^n \doteq H_n(P(x/2)),$$

from which the relation (14) follows upon our replacing x by $2x$.

Equation (14) may also be derived by applying the linear differential operator

$$B_1 = \sum_{k=0}^{\infty} \frac{(-1)^k P_k(x) D^k}{2^k k!},$$

where $D=d/dx$, to the polynomial $H_n(2x)$ in the manner of Section 5 below.

It is known [3] that the Legendre polynomials have the generating function indicated in

$$e^{tx} J_0(t\sqrt{1-x^2}) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}. \tag{30}$$

With equation (30) at hand it is easy to show by multiplication of series that

$$e^{2xt-t^2} J_0(2t\sqrt{1-x^2}) \doteq \sum_{n=0}^{\infty} H_n(P(x)) \frac{t^n}{n!}.$$

The left member of the above equation may also be obtained from the product of the series

$$e^{4xt-t^2} = \sum_{n=0}^{\infty} H_n(2x) \frac{t^n}{n!}$$

and

$$e^{-2xt} J_0(2t\sqrt{1-x^2}) = \sum_{n=0}^{\infty} (-1)^n 2^n P_n(x) \frac{t^n}{n}.$$

Thus the relation (14) may be obtained by the third method.

5. Derivation of the relation (24). Let us now consider the operator

$$B_2 = \sum_{k=0}^{\infty} \frac{(-1)^k D^{2k}}{2^k k!},$$

where $D=d/dx$. First we find that

$$B_2 x^n = \sum_{k=0}^{[n/2]} \frac{(-1)^k n! x^{n-2k}}{2^k k!(n-2k)!} = 2^{-n} H_n(x).$$

Next we apply B_2 to the polynomial $H_n(wx)$ where w is independent of x. Of course,

$$B_2H_n(wx) \doteq H_n\left(\frac{w}{2}H(x)\right),$$

but also

$$B_2H_n(wx) = \sum_{k=0}^{[n/2]} \frac{(-1)^k D^{2k} H_n(wx)}{2^{2k}k!}.$$

A familiar result on the Hermite polynomials is that

$$D^s H_n(x) = \frac{2^s n! H_{n-s}(x)}{(n-s)!}.$$

Therefore,

$$B_2H_n(wx) = \sum_{k=0}^{[n/2]} \frac{(-1)^k w^{2k} n! H_{n-2k}(wx)}{k!(n-2k)!}$$

$$\doteq w^n H_n\left(\frac{H(wx)}{2w}\right),$$

and equation (24) follows from the two formulas for $B_2H_n(wx)$.

The relation (24) may also be obtained by showing that each member of it is the coefficient of $t^n/n!$ in the expansion of the generating function $\exp[2xwt-(w^2+1)t^2]$.

6. Derivation of the relation (18). If the polynomials $\psi_n(x)$ are generated as indicated in

$$F(x, t) = \sum_{n=0}^{\infty} \psi_n(x) \frac{t^n}{(n!)^2},$$

then

$$e^t F(x, -t) = \left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n \psi_n(x) t^n}{(n!)^2}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k \psi_k(x) n!}{(k!)^2(n-k)!} \frac{t^n}{n!}$$

$$\doteq \sum_{n=0}^{\infty} L_n(\psi(x)) \frac{t^n}{n!}.$$

From Bateman [4] we borrow the fact that

$$I_0(\sqrt{2t(x-1)})I_0(\sqrt{2t(x+1)}) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{(n!)^2},$$

where $I_0(x)$ is the modified Bessel function, $I_0(iy) = J_0(y)$. Letting $P_n(x)$ play the role of the $\psi_n(x)$ above, we may write

$$e^t J_0(\sqrt{2t(x-1)})J_0(\sqrt{2t(x+1)}) \doteq \sum_{n=0}^{\infty} L_n(P(x)) \frac{t^n}{n!}. \tag{31}$$

Let us replace t by $2t$ in (31), thus arriving at

$$e^{2t}J_0(2\sqrt{t(x-1)})J_0(2\sqrt{t(x+1)}) \doteq \sum_{n=0}^{\infty} 2^n L_n(P(x))\,\frac{t^n}{n!}.$$

This reminds us of equation (3), from which we get

$$e^t J_0(2\sqrt{t(x-1)}) = \sum_{n=0}^{\infty} L_n(x-1)\,\frac{t^n}{n!}$$

and

$$e^t J_0(2\sqrt{t(x+1)}) = \sum_{n=0}^{\infty} L_n(x+1)\,\frac{t^n}{n!}.$$

Now we can conclude that

$$\sum_{n=0}^{\infty} 2^n L_n(P(x))\,\frac{t^n}{n!} \doteq \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{L_k(x-1)L_{n-k}(x+1)n!}{k!(n-k)!}\,\frac{t^n}{n!},$$

so that we have the desired relation

$$2^n L_n(P(x)) \doteq \{L(x-1)+L(x+1)\}^n. \tag{18}$$

7. Suggestions for proofs of the remaining relations. The relation (15) can be obtained from equations (11) and (29). It can be obtained by applying the operation B_1 above to $H_n(x)$. Finally, the generating function $\exp(xt-t^2)J_0(t\sqrt{1-x^2})$ may be expanded in powers of t in two ways, thus leading to equation (15).

Relations (16) and (20) can be derived simultaneously by proper use of the generating function $\exp(-t^2)J_0(2t\sqrt{1-x^2})$. In the same way (17) and (21) may be obtained from the generating function $\exp(-t^2)J_0(t\sqrt{1-4x^2})$. Equations (16) and (17) may also be found by using (29) and

$$\{x-P(x)\}^{2n+1} \doteq 0,$$

which is a special case of a known [5] result. Equations (20) and (21) may be obtained from (29) and the known [5] result

$$\{x-P(x)\}^{2n} \doteq \frac{(2n)!(x^2-1)^n}{2^{2n}(n!)^2}.$$

Equation (19) is a special case of (27), exhibited because the Laguerre polynomials are so much better known than the more general $g_n(x)$ of equation (27).

The relation (22) may be obtained by using the operator

$$B_3 = \sum_{k=0}^{\infty} \frac{H_k(y)D^k}{2^k k!}$$

on $H_n(x)$. It also follows from the generating function $\exp[2(x+y)t-2t^2]$, or from equations (7) and (29). From equations (8) and (22) we get (23) at once.

Equation (25) is most easily obtained from the generating function. The relation (25a) is a special case, $y=\sqrt{5}/2$, of (25), a particularly simple one with an interesting emphasis on the $\sqrt{5}$.

Equation (26) is an immediate consequence of (29). It may also be derived by applying

$$B_k = \sum_{k=0}^{\infty} \frac{x^k(y-1)^k D^k}{k!}$$

to $H_n(x)$, or by using the generating function $\exp(2xyt-t^2)$.

We may use any of the three methods of this paper, together with formula (13) of [2], to obtain the relation (27) above. For relation (28) we employ the same methods with the added use of formula (31) of [2].

REFERENCES

1. For (7) see E. Feldheim, Quelques nouvelles relations pour les polynômes d'Hermite, Journ. Lond. Math. Soc., 13, 1938, 22–29. For (8) see G. Szegö, Orthogonal Polynomials, Colloquium Lectures, 1939 (5. 5.11). For (9) and (10) see E. Feldheim, Développements en série de polynômes d'Hermite et de Laguerre à l'aide des transformations de Gauss et Hankel, I et II, Nederl. Akad. Wetensch., Proc., 43, 1940, 224–248. Equations (11) and (12) are special cases of formula (8) of E. D. Rainville, Notes on Legendre polynomials, Bull. Amer. Math. Soc., 51, 1945, 268–271. For (13) see E. D. Rainville, A relation between Jacobi and Laguerre polynomials, Bull. Amer. Math. Soc., 51, 1945, 266–267.

2. E. D. Rainville, Certain generating functions and associated polynomials. Amer. Math. Monthly, 52, 1945, 239–250.

3. Our (30) is used by F. H. Jackson, Proc. Lond. Math. Soc., (2), 3, 1905, 1–20. See his page 7.

4. H. Bateman, A generalization of the Legendre polynomial, Proc. Lond. Math. Soc., (2), 3, 1905, 111–123. See his formula (8).

5. In equation (8) of Notes on Legendre polynomials mentioned in reference [1] above, put $\alpha=\pi/2$ and $\cos\beta=x$.

DIFFERENTIAL GEOMETRY IN THE KASNER PLANE*

JOHN DE CICCO, Illinois Institute of Technology

1. Horn angles. A horn angle is defined as the configuration formed by two curves which pass through a point (the vertex) in a common direction. From ancient times, the qualitative aspect and quantitative nature of a horn angle had been a subject of serious consideration. Newton thought that the measure of a horn angle (of the first category) should depend on the curvatures of the two curves, calculated at the vertex.

In the nineteenth century, horn angles were revived as a concrete illustration of geometries that do not obey the Archimedian Axiom. It was noticed that different types of horn angle depending on order of contact had different qualities but it could not be decided what kind of suitable measure should be attached to each type.

* Presented to the American Mathematical Society, 1945.

In 1909, Kasner began to consider this question; the first paper appeared in the Proceedings of the Cambridge Congress of Mathematicians 1912. In his discussions from 1909–1912, Kasner showed that each category of horn angle (order of contact) has a suitable measure. This measure for each category is invariant under the total group of conformal transformations. It was proved that this invariant is essentially unique for each category.

Consider the simplest case, namely the horn angle of the first category (simple contact). This is formed by two distinct curves starting at a point in the same direction with different curvatures. The measure M_{12} of a horn angle of the first category is

$$(H) \qquad M_{12} = \frac{(\gamma_2 - \gamma_1)^2}{\left(\frac{d\gamma_2}{ds_2} - \frac{d\gamma_1}{ds_1}\right)},$$

where γ_1 and γ_2 are the curvatures of the two curves C_1 and C_2, and $d\gamma_1/ds_1$ and $d\gamma_2/ds_2$ are the rates of variation of the curvature with respect to the arc length, all of which are calculated at the vertex.

For a horn angle of category n, there is a measure M_{12} involving derivatives of order $(2n+1)$. This is invariant under the conformal group. It is essentially unique for each category. Any such invariant may serve to characterize the conformal group within the group of contact transformations of the plane.

Kasner has also studied the horn angle with respect to the equilong group which is roughly dual to the conformal group. The invariants discussed above are no longer useful but an analogous set of invariants have been found which are valid only for the equilong group.

2. The Kasner plane. In order to discuss the conformal geometry of horn angles of the first category, it is found convenient to introduce an associated plane which is called the *Kasner plane* or the *K_2-plane*. This is defined in the following manner. A simple horn-set consists of all the curves (third order differential elements) which possess a common point and a common direction. Let $x=\gamma$ denote the curvature and $y=d\gamma/ds$ the variation of curvature per unit arc length of any curve C of the simple horn-set, calculated at the common point. Any curve C (third order differential element) of the simple horn-set is given by an ordered pair of numbers (x, y). Thus a simple horn-set is a two-dimensional space which is termed the Kasner plane or the K_2-plane. Any point $P(x, y)$ of this plane, where x is the abscissa and y is the ordinate, corresponds to a curve C of the given simple horn-set where x is the curvature and y is the rate of variation of the curvature of C.

The group of conformal transformations operating on the curves of a simple horn-set induces the following three-parameter group G_3 of transformations

$$(1) \qquad X = mx + h, \qquad Y = m^2y + k,$$

where $m \neq 0$, h, k are constants, between the points (x, y) of the Kasner plane.

Thus this plane is a special type of affine plane. In preceding papers, the geometry of this *fundamental K-group* G_3 has been studied. In the present paper, we shall begin the study of the differential geometry of this group.

A *general line* consists of the ∞^1 points $P(x, y)$ which satisfy a linear equation of the form $y=px+r$, where $p\neq 0$ and r are constants. An *infinite line* (or *zero line*) is the totality of ∞^1 points $P(x, y)$ which satisfy the special linear equation $y=$const. (or $x=$const.). The infinite and zero lines are termed the *minimal lines* of the K_2-plane. Under the K-group G_3, any general line (or infinite line, or zero line) is converted into any other general line (or infinite line, or zero line).

Two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ of the K_2-plane possess the fundamental invariant

$$M_{12} = M(P_1, P_2) = \frac{(x_2 - x_1)^2}{(y_2 - y_1)}, \qquad M_{21} = -M_{12}. \tag{2}$$

This is called the *distance* M_{12} between the two points. Note that this corresponds to the measure M_{12} of the horn angle in the associated simple horn-set.

This distance is zero (or infinite) only for points along a zero line (or infinite line). It is indeterminate if and only if the two points coincide.

The distances between three points P_1, P_2, P_3, such that no two lie on the same minimal line, satisfy the triangular inequality

$$M_{12}M_{23}M_{31}(M_{12} + M_{23} + M_{31}) \leqq 0. \tag{3}$$

The equality is valid only in the case when the three points are collinear (thus obtaining a wide-open trihorn).

Two general lines $L_1: y=p_1x+r_1$ and $L_2: y=p_2x+r_2$, possess the invariant

$$\alpha_{12} = \alpha(L_1, L_2) = p_2/p_1, \qquad \alpha_{21} = 1/\alpha_{12}. \tag{4}$$

This is called the *dihorn angle* α_{12} between the two general lines. If the dihorn angle α_{12} is positive, then we define the *radian dihorn angle* θ_{12} by the formula

$$\theta_{12} = \theta(L_1, L_2) = \tfrac{1}{2} \log \alpha_{12} = \tfrac{1}{2} \log (p_2/p_1), \qquad \theta_{21} = -\theta_{12}. \tag{5}$$

3. The arc length of a curve in the Kasner plane. An arbitrary set of ∞^1 points in the K_2-plane is called a curve. This may be defined by the parametric equations $x=x(t)$, $y=y(t)$. From this it follows that a curve in the K_2-plane may be given either by the equation $y=y(x)$, or by the equation $x=x(y)$.

Of course, a line is a simple example of a curve in the K_2-plane. Another important example is a *parabolic-circle*. This may be defined as the locus of a point $P(x, y)$ which is at a fixed distance M from a fixed point $P_0(x_0, y_0)$. The point P_0 is called the center and M is termed the radius of the parabolic-circle. According to (2), the equation of any parabolic-circle is

$$(x - x_0)^2 = M(y - y_0). \tag{6}$$

By (2), it is found that the differential ds of arc length in the K_2-plane is given by

$$ds = \frac{dx^2}{dy}. \tag{7}$$

The K_2-plane is a special type of Finsler space. It is the best known example which is neither euclidean nor riemannian. The arc length is thus defined by

$$s = \int \frac{\left(\frac{dx}{dt}\right)^2}{\left(\frac{dy}{dt}\right)} dt = \int \frac{1}{\left(\frac{dy}{dx}\right)} dx = \int \left(\frac{dx}{dy}\right)^2 dy. \tag{8}$$

It follows from this that the extremals of the K_2-plane are the lines. Also transversality is obtained by taking half the slope. That is, the dihorn angle 1/2 is a right dihorn angle.

It is remarked that the arc s intercepted on a parabolic-circle of radius M by a radian-dihorn angle θ with vertex at the center, is $s = M\theta$.

4. The rac curvature R of a curve. In 1913, Kasner considered the *rac* curvature R of a curve in any space, which may be euclidean or riemannian or of the Finsler type. The *rac* curvature R at a given point P of a curve is defined to be the limit of the ratio of the arc PQ to the chord PQ as the point Q approaches P along the curve.

We shall consider the *rac* curvature R of a curve in the K_2-plane. Thus R is defined by the expression

$$R = \lim_{Q \to P} \frac{\text{arc } PQ}{\text{chord } PQ} = \lim_{(x_2, y_2) \to (x_1, y_1)} \frac{\int_{(x_1, y_1)}^{(x_2, y_2)} \frac{dx^2}{dy}}{\frac{(x_2 - x_1)^2}{(y_2 - y_1)}}. \tag{9}$$

A curve of the K_2-plane is termed an analytic element if the abscissa x and ordinate y of any point on the curve are defined by integral power series in a parameter t. Thus any analytic element is defined by

$$\begin{aligned} x &= x_0 + a_p t^p + a_{p+1} t^{p+1} + \cdots, \quad a_p \neq 0, \\ y &= y_0 + b_q t^q + b_{q+1} t^{q+1} + \cdots, \quad b_q \neq 0. \end{aligned} \tag{10}$$

Upon eliminating the parameter t, it is found that any analytic element is given by

$$y - y_0 = c_q (x - x_0)^{q/p} + c_{q+1} (x - x_0)^{(q+1)/p} + \cdots, \quad c_q \neq 0. \tag{11}$$

Under the group (1), the positive integers p and q are arithmetic invariants. Accordingly *the species* (p, q) is defined to consist of all analytic elements with fixed exponents (p, q) but with arbitrary coefficients $c_q, c_{q+1}, \cdots$. An analytic element is said to be regular if it belongs to one of the species $(1, q)$ or $(p, 1)$,

or if q/p (or p/q) is an integer and the development of y in terms of x (or x in terms of y) consists of only positive integral powers of x (or y). Otherwise it is said to be irregular.

THEOREM 1. *The rac curvature R of an analytic element of species (p, q) is given by the formula*

$$R = \frac{p^2}{q(2p - q)}. \tag{12}$$

This result may be contrasted with the corresponding one obtained by Kasner in the euclidean plane. If (x, y) are interpreted as minimal coordinates of a point so that the linear element is given by $ds^2 = dxdy$, then the *rac* curvature R of an analytic element of species (p, q) is given by the formula

$$R = \frac{2\sqrt{pq}}{p + q}. \tag{13}$$

The proof of our Theorem 1 is obtained by substituting the equations (10) defining an analytic element of species (p, q) into the formula (9) for the *rac* curvature.

From (12), we find that the ratio q/p in terms of the *rac* curvature R is given by

$$\frac{q}{p} = 1 \pm \sqrt{\frac{R - 1}{R}}. \tag{14}$$

Thus a finite number R is the *rac* curvature of an analytic element if and only if R is a non-zero rational number such that $(R^2 - R)$ is the square of a rational number. We make the following remarks.

(a) There are two distinct species of analytic elements with the given *rac* curvature $R > 1$.

(b) There is only one species, namely (p, p), of analytic elements with the given *rac* curvature $R = 1$. In this case the analytic element is tangent to neither of the minimal lines.

(c) There are no analytic elements with the given *rac* curvature R such that $0 \leqq R < 1$.

(d) There is only one species of analytic elements with the given *rac* curvature $R < 0$.

(e) For the species $(p, 2p)$, the *rac* curvature R is infinite.

5. The osculating parabolic-circles of a general curve of the K_2-plane. A curve such that its *rac* curvature R at any point is unity is termed a *general curve*. The tangent lines of a general curve are all general lines. In the neighborhood of any one of its points (x_0, y_0), a general curve may be given by $y = y(x)$ where $y(x)$ is a power series in integral powers of $(x - x_0)$ such that $y'(x_0) \neq 0$.

In the remainder of this paper, we shall study the differential geometry of general curves.

The parabolic-circle which has three consecutive points in common with the general curve $y=y(x)$ at a given point P is called the *osculating parabolic-circle* of the general curve at the point P. Its center is termed the *center of curvature* and its radius is called the *radius of curvature.*

The center of curvature (a, b) and the radius of curvature ρ of the osculating parabolic-circle at any point P of the general curve $y=y(x)$ are given by the equations

$$a = x - \frac{y'}{y''}, \qquad b = y - \frac{y'^2}{2y''}, \qquad \rho = \frac{2}{y''}. \tag{15}$$

THEOREM 2. *The necessary and sufficient condition that the* ∞^1 *parabolic-circles*

$$[x - a(t)]^2 = \rho(t)[y - b(t)], \tag{16}$$

be an osculating set of parabolic-circles is

$$a'^2 + b'\rho' = 0. \tag{17}$$

The curve to which these parabolic-circles (16) are the osculating parabolic-circles is given by the equations

$$x = a + \frac{\rho b'}{a'}, \qquad y = b + \rho \frac{b'^2}{a'^2}. \tag{18}$$

We prove our Theorem 2. The parabolic-circles (16) must be necessarily the osculating parabolic-circles of their envelope. Hence the parabolic-circles must have contact of the second order with their envelope. Therefore consecutive parabolic-circles of (16) must possess a common lineal-element. Imposing this last condition on the equations (16), we find

$$\begin{aligned} x &= a + \tfrac{1}{2}\rho p, & y &= b + \tfrac{1}{4}\rho p^2, \\ 0 &= a' + \tfrac{1}{2}\rho' p, & 0 &= b' + \tfrac{1}{4}\rho' p^2, \end{aligned} \tag{19}$$

where $p=dy/dx$ is the slope of the curve. Eliminating p from the last two equations, we obtain our condition (17). Solving the first of the last two equations for p and substituting this value of p into the first two equations, we obtain the equations (18). This completes our proof.

6. The curvature γ of a general curve. Let a general curve be given by the parametric equations $x=x(t)$, $y=y(t)$. Solving the formula (8) defining the arc length s, for the parameter t, it is seen that $t=t(s)$. Substituting this into the parametric equations of the curve, it is found that a general curve may be given by the parametric equations $x=x(s)$, $y=y(s)$, where the arc length s is the parameter. By (8), we obtain the following result.

THEOREM 3. *A general curve is given by the parametric equations* $x=x(s)$, $y=y(s)$, *where* s *is the arc length, if and only if*

$$\left(\frac{dx}{ds}\right)^2=\left(\frac{dy}{ds}\right). \tag{20}$$

The reciprocal of the radius of curvature ρ of a general curve is termed the *curvature* $\gamma=1/\rho$. From the definition of the radian dihorn angle θ, we obtain the following proposition.

THEOREM 4. *The curvature* γ *of the general curve* $y=y(x)$ *is equal to the rate of change of the radian dihorn angle that the general tangent line makes with a fixed general line, per unit arc length. The curvature* γ *is given by the equation*

$$\gamma=\frac{d\theta}{ds}=\frac{1}{2}\frac{d^2y}{dx^2}=\frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{2\left(\dfrac{dx}{dt}\right)^3}. \tag{21}$$

Now we shall prove the following fundamental theorem.

THEOREM 5. *Two general curves of the* K_2*-plane which have their curvatures* γ *the same functions of the arc length* s *are equivalent under the* K*-group* G_3. *Thus the intrinsic equation of a general curve is*

$$\gamma=\gamma(s). \tag{22}$$

Let us find all the general curves which satisfy the condition (22). By (20), (21), and (22), we find that all the curves which satisfy this condition are given by

$$\begin{aligned} x &= m\int_{s_0}^{s}\exp\left[2\int_{s_0}^{s}\gamma ds\right]ds+h \\ y &= m^2\int_{s_0}^{s}\exp\left[4\int_{s_0}^{s}\gamma ds\right]ds+k, \end{aligned} \tag{23}$$

where $m\neq 0$, h, k, are arbitrary constants. This means that as we change m, h, k, we obtain different curves which satisfy (22). But by (1), it is found that any two such curves are equivalent under the K-group G_3. This completes the proof of our Theorem 5.

7. The theory of evolutes and involutes. The locus of the centers of curvature of a given general curve is called the *evolute* of that curve. By a *normal* to a general curve at a point P, we shall mean the general line N which passes through the given point P and which is orthogonal to the general tangent line T of the curve. That is, the dihorn angle $\alpha(T, N)=1/2$.

THEOREM 6. *The envelope of the normals of a general curve is the evolute. Thus the normals of a given curve are tangent to its evolute.*

The normals of the general curve $y=y(x)$ are given by the equation

$$Y - y(x) = \tfrac{1}{2}y'(x)(X - x). \tag{24}$$

Upon finding the envelope of this family of lines, we obtain the first two of the equations (15). Thus our Theorem 6 is proved.

Let S denote the length of arc of the evolute of a given curve. Upon substituting this into (17), we see that the condition (17) may be written in the form

$$dS + d\rho = 0. \tag{25}$$

THEOREM 7. *The length of arc of the evolute is equal to the difference between the radii of curvature of the given curve which are tangent to this arc at its extremities. Also the radius of curvature of the given curve decreases as fast as the arc of the evolute increases.*

Let a curve E be given by the parametric equations $a=a(t)$, $b=b(t)$, where a and b are functions of the parameter t. Any curve I such that the given curve E is the evolute of I is called an *involute* of E. From (18) and (25), it is seen that all involutes I of the curve $E: a=a(t)$, $b=b(t)$, are given by the equations

$$\begin{aligned} x &= a - \frac{b'}{a'}\int_{t_0}^{t}\frac{a'^2}{b'}\,dt + C\frac{b'}{a'}, \\ y &= b - \frac{b'^2}{a'^2}\int_{t_0}^{t}\frac{a'^2}{b'}\,dt + C\frac{b'^2}{a'^2}, \end{aligned} \tag{26}$$

where C is an arbitrary constant. This shows that for a given curve E, there exist ∞^1 involutes.

THEOREM 8. *A given curve E of the K_2-plane possess ∞^1 involutes. These involutes are all parallel to each other.*

To finish the proof of Theorem 8, we shall consider a special type of lineal-element transformation in the K_2-plane. A *dilatation* D_k is the lineal element transformation which slides each element of the K_2-plane through a constant distance k in a direction orthogonal (that is, the dihorn angle 1/2) to itself. The equations of the dilatation D_k are

$$X = x + \tfrac{1}{2}kp, \qquad Y = y + \tfrac{1}{4}kp^2, \qquad P = p. \tag{27}$$

Every dilatation is a contact transformation. It carries every curve into a curve. We shall say that two curves of the K_2-plane are parallel if one curve can be carried into the other by a dilatation.

These considerations show that all involutes of a given curve are parallel. The proof of Theorem 8 is complete.

BIBLIOGRAPHY

1. Kasner, Conformal geometry, Proceedings of the Fifth International Congress of Mathematicians, Cambridge, Vol. 2, pp. 81–90, 1912.

2. Kasner, Conformal classification of analytic arcs or elements, Poincare's local problem of conformal geometry. Transactions of the American Mathematical Society, Vol. 16, pp. 333–349, 1915.

3. Pfeiffer, On the conformal geometry of analytic arcs, American Journal of Mathematics, Vol. 37, pp. 395–430, 1915.

4. Kasner, Equilong invariants and convergence proofs. Bulletin of the American Mathematical Society, Vol. 23, pp. 341–347, 1917.

5. Kasner, The ratio of the arc to the chord of an analytic curve need not approach unity, Bulletin of the American Mathematical Society, Vol. 20, pp. 524–531, 1914.

6. Kasner, Complex geometry and relativity: Theory of the *rac* curvature, Proceedings of the National Academy of Sciences, Vol. 18, pp. 267–274, 1932.

7. Ostrowski, Jahres. Deut. Math. Vereinigung, Vol. 44, p. 56, 1934.

8. Kasner, Conformal geometry, Science, Vol. 82, pp. 622–623, 1935.

9. Kasner and Comenetz, Conformal geometry of horn angles, Proceedings of the National Academy of Sciences, Vol. 22, pp. 303–309, 1936.

10. Kasner, Fundamental theorems of trihornometry, Science, Vol. 85, pp. 480–482, 1937.

11. Kasner, Trihornometry: A new chapter in conformal geometry, Proceedings of the National Academy of Sciences, Vol. 23, pp. 337–341, 1937.

12. Kasner, Geometry of conformal symmetry (Schwarzian reflection), Annals of Mathematics, Vol. 38, pp. 873–879, 1937.

13. Comenetz, Kasner's invariant and trihornometry, this MONTHLY, Vol. 45, pp. 82–87, 1938.

14. Kasner, Characterization of the conformal group and the equilong group by horn angles, Duke Mathematical Journal, Vol. 4, pp. 95–106, 1938.

15. Kasner, The two conformal invariants of fifth order, Transactions of the American Mathematical Society, Vol. 44, pp. 25–31, 1938.

16. Kasner and De Cicco, Conformal geometry of horn angles of second order, Proceedings of the National Academy of Sciences, Vol. 24, pp. 393–400, 1938.

17. Comenetz, Conformal geometry on a surface, Annals of Mathematics, Vol. 39, pp. 863–871, 1938.

18. Kasner and De Cicco, General trihornometry of second order, Proceedings of the National Academy of Science, Vol. 25, pp. 479–481, 1939.

19. De Cicco, The analogue of the Moebius group of circular transformations in the Kasner plane, Bulletin of the American Mathematical Society, Vol. 45, pp. 936–943, 1939.

20. De Cicco, Analog of the nine-point circle in the Kasner plane, this MONTHLY, Vol. 46, pp. 627–634, 1939.

21. Kasner and De Cicco, General invariants of irregular analytic elements, Proceedings of the National Academy of Sciences, Vol. 27, pp. 88–92, 1941.

22. Kasner and De Cicco, Conformal geometry of third order differential elements, Revista de la Universidad Nacional de Tucuman, Argentina, Vol. 2, pp. 51–58, 1941.

23. De Cicco, General comparison of conformal and equilong geometries, National Mathematics Magazine, Vol. 16, pp. 275–279, 1942.

24. Kasner and De Cicco, The general invariant theory of irregular analytic arcs or elements, Transactions of the American Mathematical Society, Vol. 51, pp. 232–254, 1942.

25. De Cicco, The magnilong near-Laguerre transformations, National Mathematics Magazine, Vol. 19, 1945.

26. De Cicco, Equilong geometry of third order differential elements, National Mathematics Magazine, Vol. 19, 1945.

27. Kasner, History of the horn angle, Scripta Matematica, 1946.

THE $\alpha+\beta$ HYPOTHESIS AND RELATED PROBLEMS

PAUL ERDÖS, Stanford University
and
IVAN NIVEN, Purdue University

1. Introduction. Let $a_1<a_2<a_3\cdots$ be a set, A, of positive integers. Let $A(n)$ denote the number of integers of A which are not greater than n. The *asymptotic density* $\delta(A)$ of A is defined to be the lower limit of $A(n)/n$. This is a natural definition of density: the set of all (positive) integers has asymptotic density 1; the odd integers, $\frac{1}{2}$; the even integers, $\frac{1}{2}$; the integers which are multiples of 7, 1/7.

A less natural definition, but one which has attracted considerable attention, is the *greatest lower bound* of $A(n)/n$. This is the *Schnirelmann density* [14] of A, denoted here by $d(A)$. The odd integers have density $\frac{1}{2}$, clearly, but the even integers have density 0 under this definition. In fact, if a set A lacks the integer 1, then $d(A)=0$. Although this may seem artificial, Schnirelmann was led to the definition quite naturally by a study of certain problems in additive number theory, as we shall see later.

It is clear from the definitions that $0\leqq d(A)\leqq\delta(A)\leqq 1$.

Let B be the set of positive integers $b_1<b_2<b_3<\cdots$. The sum $A+B$ of the sets A and B is defined as the set of integers of the form a_i or b_j or a_i+b_j; that is, an integer x is in $A+B$ if x is in A, or if x is in B, or if x is the sum of an integer of A and an integer of B. For convenience we denote $d(A)$, $d(B)$, and $d(A+B)$ by α, β, and γ, respectively. The question of the relation of γ to α and β has been the subject of much investigation, culminating in Mann's proof [12] of the celebrated Khintchine conjecture [8], commonly known as the $\alpha+\beta$ hypothesis, which is that γ is not less than the minimum of 1 and $\alpha+\beta$. This paper outlines briefly, and not at all exhaustively, the history and present state of knowledge of this problem and some related ones. Some of the simplest proofs are given.

Landau [11] and Rohrbach [13] have summarized much of the principal work in this field up to 1937 and 1938 respectively.

2. Asymptotic density. First we mention some well known sequences of integers and their asymptotic densities. If A is the set of prime numbers, then $\delta(A)=0$ [9]. The set of square-free integers has density $6/\pi^2$ [7]. If A is either the set of squares or the integers which are expressible as sums of two squares, then $\delta(A)=0$. The set of integers which can be written as sums of three or fewer squares has density 5/6, as can be readily deduced from the fact that this set contains all integers except those of the form $4^a(7+8b)$.

Davenport [4] has shown that if A represents the set of integers which are sums of s or fewer cubes of positive integers, then $\delta(A)=1$ for $s=4$ but for no smaller value of s. Change cubes to fourth powers, change $s=4$ to $s=15$, and you get another result of Davenport [5]. In each of these cases the fact that the result is best possible can be proved by simple congruences. For example, no

integer of the form $15+16a$ can be a sum of 14 fourth powers, because any fourth power is congruent to 0 or 1 (mod 16).

Less is known about the higher powers. If $A_{k,s}$ represents the set of integers which are expressible as sums of s or fewer kth powers of positive integers, it is known that $\delta(A_{k,k-1})=0$; but whether $\delta(A_{k,k})$ is zero or positive is not known for $k\geqq 3$.

We mentioned in §1 the problem of the relationship between the Schnirelmann densities of A, B, and $A+B$. For asymptotic densities the $\alpha+\beta$ hypothesis is not true: consider the case, for example, where both A and B are composed of all positive even integers, so that $A+B$ is the same set, and we have $\delta(A)=\delta(B)=\delta(A+B)=\frac{1}{2}$.

However, Erdös [6] has proved the following result. Let α', β', and γ' stand for $\delta(A)$, $\delta(B)$, and $\delta(A+B)$. If $\gamma'\leqq 1$, $\alpha'\leqq\beta'$, and $a_1=1$ (recall that a_1 is the first integer of A), then $\gamma'\geqq\frac{1}{2}\alpha'+\beta'$. This result is best possible in the sense that sets A and B can be exhibited so that the equality sign holds in the conclusion. This can be done by taking A and B to be the same set, the set of integers which are congruent to 0 or 1 (mod 4), so that $A+B$ is composed of all integers congruent to 0, 1, or 2 (mod 4).

3. Two proofs. Henceforth we shall discuss only Schnirelmann density. Schnirelmann [14] and Landau [10] proved that $\gamma\geqq\alpha+\beta-\alpha\beta$. We now prove this result.

Proof. Consider any integer a in A such that $a+1$ is not in A. Let the next largest integer in A be $a+h+1$. Thus we have a gap of length h, that is, a gap of exactly h integers which are not in A. Consider now any integer x, not necessarily in A. Let there be m gaps of lengths $h_1, h_2, \cdots, h_m$ in A among the integers which are $\leqq x$. In case x is not in A, we shall take h_m to be the number of consecutive integers $x, x-1, x-2, \cdots$ which are not in A; thus h_m, unlike the other h's, may not represent the length of a *complete* gap in the integers of A.

Now since $A(x)$ denotes the number of integers in A which are $\leqq x$, we have

$$x - A(x) = h_1 + h_2 + \cdots + h_m. \tag{1}$$

Moreover, a and $a+h+1$ being in A (but no integer between these two), we can add the $B(h)$ integers of B which are $\leqq h$ to a to get $B(h)$ integers in $A+B$ between a and $a+h+1$. Doing this for all the gaps of lengths $h_1, h_2, \cdots, h_m$ we see that, denoting $A+B$ by C,

$$C(x) \geqq A(x) + B(h_1) + B(h_2) + \cdots + B(h_m). \tag{2}$$

But by definition $B(h)\geqq\beta h$, and hence (2) implies

$$C(x) \geqq A(x) + \beta(h_1 + h_2 + \cdots + h_m),$$

which combines with (1) to give

$$C(x) \geqq A(x) + \beta(x - A(x)) = (1 - \beta)A(x) + \beta x.$$

But $A(x) \geqq \alpha x$, so we have

$$C(x) \geqq (1-\beta)\alpha x + \beta x = x(\alpha+\beta-\alpha\beta),$$

which completes the proof.

As we have stated the $\alpha+\beta$ hypothesis it includes the result that if $\alpha+\beta \geqq 1$, then $\gamma=1$. This is very simple, and we prove it now.

Proof. We must prove that $A+B$ is the set of all integers. Suppose a certain integer n is not in $A+B$. Let $a_1<a_2<a_3<\cdots<a_k$ be the members of A which are $<n$. Then $A(n)=k$ and $k/n=A(n)/n \geqq \alpha$. Also it is clear that B lacks the $k+1$ integers $n-a_1, n-a_2, \cdots, n-a_k, n$. Hence β, the density of B, is at most $1-(k+1)/n$. Thus we have

$$\alpha \leqq \frac{k}{n}, \qquad \beta \leqq 1-\frac{k+1}{n}, \qquad \alpha+\beta \leqq 1-\frac{1}{n},$$

which contradicts the hypothesis.

4. Concerning Goldbach's hypothesis. Schnirelmann [14] proved that if A represents the set of all primes (we include 1 as a prime for convenience in this discussion), then, although $d(A)=0$, $d(A+A)$ is positive. By the Schnirelmann-Landau theorem of §3 (or a fortiori by the $\alpha+\beta$ hypothesis) it follows from $d(A+A)=d(2A)=\lambda>0$ that $d(4A) \geqq \lambda+\lambda-\lambda^2$ which exceeds $\frac{3}{2}\lambda$ if $\lambda<\frac{1}{2}$. By induction we know that there exists an n such that $d(nA)>\frac{1}{2}$ and the second result proved in §3 implies that $d(2nA)=1$. Thus Schnirelmann was able to show the existence of a constant $2n$ such that every integer is expressible as a sum of $2n$ primes. Later, Vinogradoff proved by different methods that every sufficiently large odd integer is a sum of three primes. Goldbach's hypothesis is that every even integer is a sum of two primes.

5. The $\alpha+\beta$ hypothesis. Khintchine [8] conjectured the $\alpha+\beta$ hypothesis (that $\gamma \geqq \alpha+\beta$ or $\gamma=1$, a best possible result), having proved it in the special cases $\alpha=\beta$ and $\alpha=1-2\beta$. There followed a series of papers proving partial or modified results.

The Schnirelmann-Landau result has already been mentioned in §3. Besicovitch [2] defined β^* as the greatest lower bound of $B(n)/(n+1)$, and proved that the density of integers of the form a_i or a_i+b_j is not less than $\alpha+\beta^*$, a result which is best possible.

Schur [15] proved that $\gamma \geqq \alpha/(1-\beta)$ or $\gamma=1$. Brauer [3] proved that $\gamma \geqq \frac{9}{10}(\alpha+\beta)$ or $\gamma=1$.

Landau [11, p. 7] posed the question of the as yet undecided problem and wrote: "Ich weiss es nicht: dies ungelöste Problem möchte ich dem Leser ans Herz legen."

Finally Mann [12] became interested in the problem while in attendance at a series of lectures on number theory by A. T. Brauer, and achieved the result that had eluded so many, the proof of the $\alpha+\beta$ hypothesis. Later Artin and Scherk [1] gave a simplification of Mann's proof.

REFERENCES

1. E. Artin and P. Scherk, On the sum of two sets of integers, Annals of Math., vol. 44, 1943, pp. 138–142.
2. A. S. Besicovitch, On the density of the sum of two sequences of integers, Journal of the London Math. Soc., vol. 10, 1935, pp. 246–248.
3. A. Brauer, On the density of the sum of sets of positive integers II, Annals of Math., vol. 42, 1941, pp. 959–988.
4. H. Davenport, On Waring's problem for cubes, Acta Math. 71, 1939, pp. 123–143.
5. H. Davenport, On Waring's problem for fourth powers, Annals of Math., vol. 40, 1939, pp. 731–747.
6. P. Erdös, On the asymptotic density of the sum of two sequences, Annals of Math., vol. 43, 1942, pp. 65–68.
7. G. H. Hardy and E. M. Wright, The theory of numbers. Oxford Press, 1938, p. 268.
8. A. Khintchine, Zur additiven Zahlentheorie, Recueil math. de la soc. math. Moscou, 39, 1932, pp. 27–34.
9. E. Landau, Handbuch der Primzahlen, vol. 1, 1909, p. 69.
10. E. Landau, Die Goldbachsche Vermutung und der Schnirelmannsche Satz, Nachrichten Göttingen, 1930, pp. 255–276.
11. E. Landau, Über einige neuere Fortschritte der additiven Zahlentheorie, Cambridge Tract 35, 1937, Chapters 2–4.
12. H. B. Mann, A proof of the fundamental theorem on the density of sums of sets of positive integers, Annals of Math., vol. 43, 1942, pp. 523–527.
13. H. Rohrbach, Einige neuere Untersuchungen über die Dichte in der additiven Zahlentheorie, Jahresber. Deutsch. Math. Verein., 48, 1938, pp. 199–236.
14. L. Schnirelmann, Über additive Eigenschaften von Zahlen, Math. Annalen, 107, 1933, pp. 649–690.
15. I. Schur, Über den Begriff der Dichte in der additiven Zahlentheorie, Sitzungs. der preuss. Akad. der Wiss., Math. Phys. Klasse, 1936, pp. 269–297.

ELLIPTIC FERMAGORIC TRIANGLES

P. A. PIZÁ, San Juan, Puerto Rico

1. Introduction. Let us consider the following numerical relations:

$$12 + 3 = 15. \qquad 12^2 + 5^2 = 13^2. \tag{1}$$

$$12^3 = (9 + \sqrt{5})^3 + (9 - \sqrt{5})^3.$$

$$12^3 + [9 - (432 + \sqrt{718065})^{1/3} - (432 - \sqrt{718065})^{1/3}]^3 = [9 + (432 + \sqrt{718065})^{1/3} + (432 - \sqrt{718065})^{1/3}]^3.$$

$$12^4 = [9 + (\sqrt{62856} - 243)^{1/2}]^4 + [9 - (\sqrt{62856} - 243)^{1/2}]^4.$$

$$12^4 + [9 - (144 + \sqrt{40419})^{1/3} - (144 - \sqrt{40419})^{1/3}]^4 = [9 + (144 + \sqrt{40419})^{1/3} + (144 - \sqrt{40419})^{1/3}]^4.$$

$$12^5 = [9 + (\sqrt{200340}/5 - 81)^{1/2}]^5 + [9 - (\sqrt{200340}/5 - 81)^{1/2}]^5.$$

These identities represent isoperimetric fermagoric triangles of constant perimeter 30, each of them sharing a common side 12. The first linear one may

be considered as a triangle of altitude zero and fermagoric order one.

If we take an ellipse of eccentricity 2/3, with major axis 18 and focal distance 12, and imagine the common leg 12 of each of these triangles placed on the segment between the foci, then their third vertex in every case will be located at a different point on the ellipse.

For this reason we have named such a family of isoperimetric fermagoric triangles, when their legs a, b, c satisfy the Fermat equation $a^n+b^n=c^n$, when they have one constant leg b or c, and when their perimeter $a+b+c$ is also constant, *elliptic fermagoric triangles.*

The generalized study of elliptic fermagoric triangles which we propose to initiate in this article, is possibly new and we shall see that it turns out to be interesting and instructive.

2. Elliptic triangles. In Fig. 1, let L be an ellipse of eccentricity $s/r<1$, where the major axis is $gh=2r^2$ and the focal distance is $ab=2rs$. Let the parameters r and s be two relatively prime integers of different parity, that is, not both odd. (It will follow later that our selection of the integer parameters r and s must be curiously restricted by certain necessary limitations.)

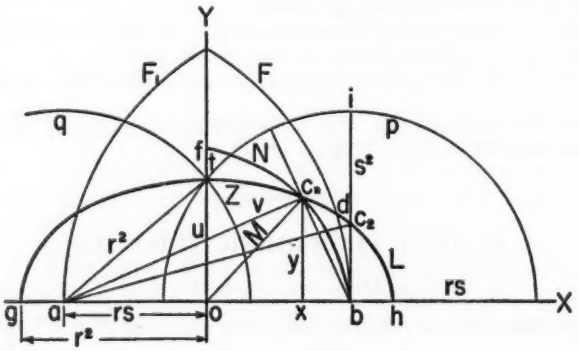

FIG. 1

In order to obtain families of isoperimetric triangles with constant perimeter $2rs+2r^2$ and each sharing a common leg $2rs$, it will suffice that their base be on the focal segment ab, and that their third vertex c_n be any point on the ellipse L.

If now we draw with radii r^2 and with centers at the foci a and b, the two focal circles p and q, these two circles intersect at the upper central point t of the ellipse. Then the two upper legs of our triangles are $r^2 \pm v$, where v is the distance from c_n to each of the circles p and q, inasmuch as the ellipse L is the locus of all points equidistant from the two focal circles p and q.

So far we are dealing with an infinite family of elliptic triangles having one common leg $ab=2rs$ within the foci, and having the constant perimeter

$$2rs + (r^2 + v) + (r^2 - v) = 2rs + 2r^2 = 2r(r + s).$$

3. Elliptic fermagoric triangles. Let us now further require that the triangles be fermagoric of successive integral orders $n>0$.

For $n=1$, we shall have $ab+bh=ah$, or $2rs+(r^2-v)=(r^2+v)$, whence $v=rs$.

Now on the focus b draw the perpendicular to gh, (latus rectum), cutting the ellipse at point c_2 and circle p at point i. Then in the right triangle abc_2 we have $(2rs)^2+(r^2-v)^2=(r^2+v)^2$. Again solving for v we obtain $v=s^2$, hence $c_2i=s^2$, and in the triangle abc_2

$$(2rs)^2 + (r^2 - s^2)^2 = (r^2 + s^2)^2,$$

which is the classical Pythagorean equation, primitive if we make the parameters r and s relatively prime integers not both odd. We see now why for convenience we did so, and why we took the focal distance and the major axis of the ellipse L to be respectively $2rs$ and $2r^2$.

In order for us to continue finding fermagoric elliptic triangles of order n greater than 2, we will have to solve for v in terms of our parameters r and s, the respective Fermat equations

Type I. $(2rs)^n+(r^2-v)^n=(r^2+v)^n$,

Type II. $(2rs)^n=(r^2-v)^n+(r^2+v)^n$,

depending on whether (r^2+v) or $2rs$ is to be the generalized fermagoric hypotenuse of the triangle that we are seeking.

We have seen that whenever r and s are integers, the type I Fermat equation renders us integral values for $v_1=rs$, $v_2=s^2$.

It is obvious that if ever an elliptic fermagoric triangle of finite integral order n greater than 2 should exist having integral and primitive sides obtainable by means of proper selection of the parameters r and s, it would have to belong to one or the other of our two types, and that the value of v would have to be integral, for if c is even, a and b must be odd, and $v=(b-a)/2$ is an integer.

The numerical identities with which we started this article were obtained by the use of the above equations making $r=3$, $s=2$. Those corresponding to type II for $n=6$, 7, 8 and 9 can also be calculated with increasingly complicated finite numerical values for v under radicals.

For the time being we shall limit our generalized study of elliptic fermagoric triangles to the equations and triangles of type II and of odd fermagoric order $n=2m+1$, which of course include all prime exponents greater than 2. We will see later that every triangle of type I, if ever its three sides are possibly rational, may be transformed into one of type II of the same fermagoric order by a convenient change in the value of the parameters.

Notice now that all elliptic fermagoric triangles of type II must exist within the fermatic region bounded by the circular arcs F_1 and F.*

One shows easily that $(oY)^2=3r^2s^2$, $(ot)^2=r^4-r^2s^2$; hence for the existence of

* See the author's 'Fermagoric Triangles' (G. E. Stechert Co., N.Y.). This paper constitutes a kind of appendixed tenth chapter to that book, and must be read with a knowledge of the book.

fermagoric triangles of type II on the ellipse, it is necessary that $oY > ot$, which is equivalent to $r < 2s$, $s/r > 1/2$ or or $4 > (r/s)^2$.

We can now determine the least value of the exponent n for which such triangles exist, by computing the real fermagoric exponent for the triangle at point t where $v = 0$ and

$$(2rs)^n = (r^2)^n + (r^2)^n, \qquad (2s/r)^n = 2,$$

the result being

$$n = \log 2/(\log 2s - \log r).$$

For $n = 3$, the solution for v of the Fermat equation

$$(2rs)^3 = (r^2 + v)^3 + (r^2 - v)^3,$$

gives us, for example, the parametric identity

$$(2rs)^3 = [r^2 + \sqrt{r(4s^3 - r^3)/3}]^3 + [r^2 - \sqrt{r(4s^3 - r^3)/3}]^3,$$

where the value of v is an irrational function of the parameters.

We can generalize the fermatic equation of type II for all odd fermagoric exponents $n = 2m+1$ greater than 2, in the following convenient manner:

(2)
$$\begin{aligned} 4^m r^n s^n - r^{2n} &= r^n(4^m s^n - r^n) \\ &= C_2^n (r^2)^{n-2}(v^2) \\ &\quad + C_4^n (r^2)^{n-4}(v^2)^2 \\ &\quad + C_6^n (r^2)^{n-6}(v^2)^3 \\ &\quad + \cdots + n r^2 (v^2)^m. \end{aligned}$$

Observe that for any odd fermagoric exponent $n = 2m+1$ greater than 2, the resulting fermatic equation (2) is of algebraic degree $m = (n-1)/2$ in v^2 as the unknown. It is solvable for real values of v in terms of radicals involving the parameters r and s, by elementary methods, when $n = 1$, 3, 5, 7 and 9. (Type II equations are also solvable for v in terms of the parameters in radicals for the even fermagoric exponents $n = 2$, 4, 6 and 8.)

For odd $n = 2m+1$ greater than 9 we shall be faced whith the solution of algebraic equations in v^2 as the unknown, of algebraic degrees $m = 5$ and higher. But remark that no matter how high be the algebraic degree m of the equation to be solved, its real root v^2 shall always be measured by the square on a vector $\mathbf{v}$ separating the upper vertex of the corresponding elliptic fermagoric triangle from the focal circles p and q.

Inasmuch as at the frontier F of the fermatic region the fermagoric exponent n is always infinite, the value of n increases on the ellipse from $n = \log 2/(\log 2s - \log r)$ at point t, to $n = \infty$ at point d, thus rendering interior triangles of type II. On the other hand the exterior triangles of type I start with exponent $n = 1$ at point h, increasing to $n = 2$ at point c_2 and finally again reaching ∞ at point d.

It is apparent that the fermagoric exponent n must be a hidden function of the length of the elliptic arc $tc_n = Z$. The writer invites collaboration in seek-

ing the secreted relation involved which so far has eluded his efforts to find, but the double periodicity of the fermagoric exponent n on the elliptic quadrant suggests a possible connection with elliptic functions.

The fact remains that there is a one to one correspondence between the points on the elliptic arc td within the fermatic region, and every real fermagoric exponent from the minimum $\log 2/(\log 2s-\log r)$ to infinity.

Let us note that in (2.) in order for v^2 to be positive we must have $4^m s^n - r^n$ positive. Whence a further necessary condition in the selection of our parameters r and s is that

$$4^m s^n > r^n. \text{ Hence } 4 > \left(\frac{r}{s}\right)^{n/m},$$

which is a more subtle requirement than the previously found inequality

$$4 > \left(\frac{r}{s}\right)^2, \text{ for } \frac{n}{m} = \frac{2n}{n-1} > 2.$$

This more precise inequality requirement also follows from the necessity that $ot < of$, where f is the Y intercept of the $n-$fermat N and $of = \sqrt{4^{(n-1)/n} - 1}\, rs$.

4. Geometry of elliptic triangles. Let x and y be the cartesian coordinates of the vertex c_n of any elliptic triangle, when the coordinate axes are the axes of the ellipse. Then

$$y^2 = (r^2 + v)^2 - (rs + x)^2$$
$$y^2 = (r^2 - v)^2 - (rs - x)^2.$$

By subtraction we get the following relation:

$$\frac{v}{x} = \frac{s}{r} = e. \tag{3}$$

By addition we obtain

$$r^2 s^2 + x^2 + y^2 = r^4 + v^2 = r^2 s^2 + M^2, \tag{4}$$

where M is the median oc_n.

By (3) we can now write the general Fermat equation of type II as follows:

$$(2rs)^n = (r^2 + ex)^n + (r^2 - ex)^n.$$

Therefore if ever this equation is possible in integers for $n > 2$ when $e = s/r$ is rational, the abscissa x of the c_n vertex of the corresponding elliptic triangle must be rational and such that $v = ex = sx/r$ is an integer.

Relation (4) proves the following theorem:

In the triangle aoc_n, whose area is half that of the elliptic triangle abc_n, the square on its base $ao = rs$, plus the square on its side $oc_n = M$, is equal to the sum of the squares on each of the two segments r^2 and v of its third side $ac_n = r^2 + v$. This relation becomes the Pythagorean theorem when $v = 0$, and the triangle is then the right triangle aot.

Many other interesting relations in elliptic triangles can be derived from the geometry of the ellipse. For instance, if ac_n is considered divided by the Y axis into two segments au and $uc_n=w$, then $w/v=(r^2+v)/(s^2+v)$ which is equivalent to $2w=(r^2+v)(1-(s^2-v)/(s^2+v))$, and to

$$(r^2+2v-w)^2=r^4+2(2s^2-r^2)w+w^2.$$

The ordinate y of the c_n vertex is

$$y=\sqrt{r^2-s^2}\sqrt{r^2s^2-v^2}/s=\sqrt{r^2-s^2}\sqrt{r^2-\left(\frac{v}{s}\right)^2},$$

which is rational when $v=s^2$ and $n=2$.

5. Reduction of case I to case II. Before we conclude, let us go back to the exterior elliptic fermagoric triangles of type I, when the vertex c_n is located on the elliptic arc dc_2 outside of the fermatic region, and (r^2+v) is the fermagoric hypotenuse.

In Fig. 2 let abc_n be such a triangle of order n greater than 2. Through point e where its longer side crosses the frontier arc F of the fermatic region, draw ef parallel to c_nb. Then triangle afe is similar to triangle abc_n and therefore it is also

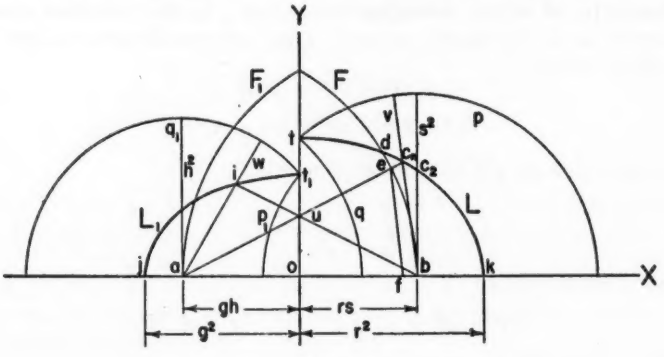

FIG. 2

fermagoric of nth order. Let u be the point where ac_n crosses the ordinate axis OY. Through b and u draw a line and on it make $bi=af$. Then triangle abi is equal to triangle aef, except that now the hypotenuse (longer side) lies on ab, and its upper vertex i rests within the fermatic region. In it

$$\frac{ai}{r^2-v}=\frac{ab}{r^2+v}=\frac{ib}{2rs},$$

where $ab=2rs$. Hence

$$ib=\frac{4r^2s^2}{r^2+v},\qquad ai=\frac{2rs(r^2-v)}{r^2+v}$$

and we have $(2rs)^n=[4r^2s^2/(r^2+v)]^n+[2rs(r^2-v)/(r^2+v)]^n$.

We may ask now: To what family of interior elliptic fermagoric triangles would the triangle abi belong?

To find out we must draw through point i and with focal distance ab, the second ellipse L_1 which will cut the Y axis at t_1 and the X axis at j. If then we make $oj=at_1=g^2$, and $ao=rs=gh$, the triangle abi will belong to the elliptic family $(2gh)^n=(g^2+w)^n+(g^2-w)^n$, which is of type II. We shall now show that if ever abc_n has rational sides, so will abi have. Therefore we can treat abi in the same way that we have treated those belonging to type II.

In the interior triangle $gh=rs$.

$$ib = g^2 + w = 4r^2s^2/(r^2 + v).$$

$$ia = g^2 - w = 2rs(r^2 - v)/(r^2 + v).$$

$$g^2 = rs[2rs + (r^2 - v)]/(r^2 + v).$$

$$w = rs[2rs - (r^2 - v)]/(r^2 + v).$$

Therefore if for some high value of n in the exterior triangle the root-vector $\mathbf{v}$ is ever rational, so will the values of gh, g^2, and w be in the similar interior triangle. The new parameters in the latter must be such that r, g, h and s diminish in value in that order. The eccentricity of the new ellipse L_1 is h/g, which must be greater than that of the first ellipse s/r.

The fact that gh, g^2 and w in $(2gh)^n=(g^2+w)^n+(g^2-w)^n$ would be rational, making this interior Fermat equation rational, does not imply that g is necessarily rational, nor that h and w are integers.

However, in our initial treatment of elliptic fermagoric triangles we chose to make the Fermat equation $(2rs)^n+(r^2-v)^n=(r^2+v)^n$ only for the convenience of getting $v=s^2$ in the case $n=2$, so that the resulting identity $(2rs)^2+(r^2-s^2)^2=(r^2+s^2)^2$ would result immediately in this classical Pythagorean solution. For either case we could just as well have started with $(2s)^n=(r+v_1)^n+(r-v_1)^n$ where $c=2s$, $(a+b)/2=r$, $(b-a)/2=v_1$, and $c^n=b^n+a^n$.

The eccentricity of the fermatic ellipse for case I would remain as before s/r, and the corresponding reduction to case II would be now $(2h)^n=(g+w_1)^n+(g-w_1)^n$ where h and g and w_1 would be rational if r, s and v_1 were integral. Now, if h and g were both rational, the eccentricity of the new ellipse L_1 would again be rational.

6. Conclusion. In this very preliminary study of the isoperimetric elliptic fermagoric triangles, we believe to have established that they are worthy of further study which we foresee may possibly be the source of new and interesting aspects of Number Theory as well as of Theory of Equations.

Their application to the study of Fermat's Problem depends on our finding more precise properties in those points on the ellipse corresponding to the apexes of fermagoric triangles of the higher orders, where the ellipse intersects the corresponding n-fermat ovals.

Perhaps we can develop relevant relations on these apex points and the Dedekind lattice of rational points on the plane, which may aid us in determining the possible rationality or irrationality of $r^2\pm v$ when $n>2$.

CONCERNING PEDAL CIRCLES AND SPHERES*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. Introduction. In the *Nouvelles Annales de Mathématiques* (1904, p. 400) M. T. Lemoyne announced the following theorem: *The radical axes of all pairs of the pedal circles, relative to a triangle ABC, of the points of a transversal Δ, are concurrent in a point ω.*

Analytical demonstrations† and geometrical solutions‡ of Lemoyne's theorem have occupied a number of geometers. We are unaware if the MONTHLY has published any of the demonstrations of this important proposition. The following, which uses only elementary geometrical properties, possesses the advantage of leading immediately to a theorem on pedal spheres, relative to a tetrahedron, of points on a line. A statement of this latter theorem was communicated to Cl. Servais (December, 1921) after having ourselves established a particular interesting case.§ The analytical and geometrical demonstrations of this space theorem as supplied by Cl. Servais are not simple.‖

2. Geometrical demonstration of the theorem of M. T. Lemoyne. Let $A_1B_1C_1$, $A_2B_2C_2$ be the pedal triangles, relative to a triangle ABC, of two points P_1, P_2 of any given line Δ, and let $A_1'B_1'C_1'$, $A_2'B_2'C_2'$ be their anticomplementary triangles.** Also, let D_1, D_2, D_3 be the points of intersection of the corresponding sides of triangles $A_1B_1C_1$ and $A_2B_2C_2$, and E_1, E_2, E_3 those of the corresponding sides of triangles $A_1'B_1'C_1'$ and $A_2'B_2'C_2'$. The lines D_1E_1, D_2E_2, D_3E_3 are concurrent in a point P, the *equicenter* of triangles $A_1B_1C_1$ and $A_2B_2C_2$.†† If P_1 remains fixed while P_2 describes the line Δ, the points A_2, B_2, C_2 describe similar ranges on BC, CA, AB, and the point P of intersection of the lines D_1E_1, D_2E_2, D_3E_3 remains fixed and has the same barycentric coordinates α, β, γ for the triangle $A_1B_1C_1$ and all the triangles $A_2B_2C_2$. Therefore, if through P one draws segments PA_3', PB_3', PC_3' equal and parallel to A_1A_2, B_1B_2, C_1C_2, the point P

* Received in April, 1942. Translated from the French by Howard Eves.

† J. Neuberg, Bull. de l'Acad. royale de Belgique, 1910; Liénard, Mathesis, 1912, p. 238 and 1914, p. 33; R. Goormaghtigh, Mathesis, 1925, p. 196; R. Goormaghtigh, The Tohoku Mathematical Journal, 1926, p. 80.

‡ de Lépiney, Mathesis, 1914, p. 177; Gallatly, The Modern Geometry of the Triangle, 2nd ed., p. 51; R. Bouvaist, Nouvelles Annales, 1915, p. 555; V. Thébault, Nouvelles Annales, 1918, p. 293; V. Thébault, The Tohoku Mathematical Journal, vol. 19, p. 33.

§ V. Thébault, The Tohoku Mathematical Journal, 1919 and 1921.

‖ Cl. Servais, Bull. de l'Acad. royale de Belgique, 1922, p. 32; Cl. Servais, Mathesis, 1922, p. 87.

** Triangle $A'B'C'$ is *anticomplementary* to triangle ABC if A, B, C are the midpoints of $B'C'$, $C'A'$, $A'B'$ respectively. Tetrahedron $A'B'C'D'$ is *anticomplementary* to tetrahedron $ABCD$ if A, B, C, D are the centroids of the faces $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$ respectively. (H. Eves)

†† E. Duporcq, Intermédiaire des Mathématiciens, 1899, p. 98; Sollerstinsky, Mem. de la Soc. royale des Sciences de Liège, 3rd series, t. X, 1913.

It may be shown that the barycentric coordinates of P with respect to triangle $A_1B_1C_1$ are the same as those with respect to triangle $A_2B_2C_2$. Hence the name *equicenter*. (H. Eves)

is the barycenter of the points A_3', B_3', C_3' for the coefficients α, β, γ.* These coefficients are, then, proportional to the products $(B_1B_2)(C_1C_2)\sin A, \cdots$, that is, to $a \sec \theta_a$, $b \sec \theta_b$, $c \sec \theta_c$,† where a, b, c are the lengths of the sides BC, CA, AB, and $\theta_a, \theta_b, \theta_c$ the angles of like sense which these sides make with Δ.

Designating by (ω_1, ρ_1), (ω_2, ρ_2) the circles with centers ω_1, ω_2 and radii ρ_1, ρ_2, circumscribed about the triangles $A_1B_1C_1$, $A_2B_2C_2$, we have‡

$$\rho_1^2 \sum \alpha = (\omega_1 P)^2 \sum \alpha + \sum (PA_1)^2 \alpha,$$
$$\rho_2^2 \sum \alpha = (\omega_2 P)^2 \sum \alpha + \sum (PA_2)^2 \alpha.$$

Also, if M_a is the midpoint of A_1A_2, P_a the projection of P on A_1A_2, we find by simple geometry that

$$\sum (PA_1^2 - PA_2^2)\alpha = \sum (A_1A_2)(M_aP_a)\alpha.$$

On the other hand let M be the midpoint of P_1P_2, which projects into M_a on A_1A_2 and into M_a' on PA_3'. We have $M_aP_a = M_a'P$ and, therefore,

$$\sum (A_1A_2)(M_aP_a)\alpha = \sum (PA_3')(M_a'P)\alpha.$$

Now, being given any n points whatever, $A_1, A_2, \cdots, A_n$, and the barycenter, G, of these points for coefficients $\alpha_1, \alpha_2, \cdots, \alpha_n$, and if $l_1, l_2, \cdots, l_n$ are the projections of any vector whatever on $GA_1, GA_2, \cdots, GA_n$, then we have, in both magnitude and sign,§

$$\sum \alpha_1 (GA_1) l_1 = 0.$$

We then have

$$\sum (PA_3')(M_a'P)\alpha = 0,$$

and therefore

$$(\omega_1 P)^2 - \rho_1^2 = (\omega_2 P)^2 - \rho_2^2,$$

which proves that P is the center of a circle (P, σ) orthogonal to circle $A_1B_1C_1$ and all the circles $A_2B_2C_2$. This proves Lemoyne's theorem.

For the square of the radius of the circle (P, σ) we have

$$\sigma^2 = (a_1^2\beta\gamma + b_1^2\gamma\alpha + c_1^2\alpha\beta)/(\alpha + \beta + \gamma)^2,$$

a_1, b_1, c_1 being the lengths B_1C_1, C_1A_1, A_1B_1. One can verify that the power of P with respect to the circles $A_1B_1C_1$, $A_2B_2C_2$ is equal to $-2d\delta$, d and δ being the distances of the circumcenter of ABC and of P from Δ, so that P coincides with the orthopole of Δ with respect to triangle ABC.

---

* See, *e.g.*, exercise 8 on page 94 of M'Clelland's Geometry of the Circle. (H. Eves)

† $[(B_1B_2)(C_1C_2)\sin A, \cdots] = [(B_1B_2)(C_1C_2)a, \cdots] = [P_1P_2 \cos\theta_a P_1P_2 \cos\theta_a a, \cdots]$
$= [a \sec \theta_a, \cdots]$. (H. Eves)

‡ See, *e.g.*, art. 55, p. 99, of M'Clelland's Geometry of the Circle. In this article replace P and O by ω_1 and P respectively. (H. Eves)

§ R. Blanchard, Journal de Vuibert, t. 50, p. 12.

3. Geometrical demonstration of the theorem of Servais and Thébault. *The pedal spheres, relative to a tetrahedron $T \equiv ABCD$, of the points of a given line Δ, have a common orthogonal sphere.*

Let $T_1 \equiv A_1B_1C_1D_1$, $T_2 \equiv A_2B_2C_2D_2$ be the pedal tetrahedra, relative to tetrahedron T, of the points P_1, P_2 on an arbitrary line Δ. If we designate by a, b, c, d the lines of intersection of the corresponding faces of the two tetrahedra T_1 and T_2, and by a', b', c', d' the lines of intersection of the corresponding faces of the tetrahedra anticomplementary to T_1 and T_2, then the planes (a, a'), (b, b'), (c, c'), (d, d') are copunctual at a point P which has the same barycentric coordinates for the tetrahedra T_1 and T_2, and the point P remains fixed when, with P_1 fixed, P_2 describes the line Δ. This point P is the *equicenter* of the tetrahedra T_1 and T_2.* If through P one draws the segments PA_3', PB_3', PC_3', PD_3' equal and parallel respectively to A_1A_2, B_1B_2, C_1C_2, D_1D_2, P is the barycenter of the points A_3', B_3', C_3', D_3' for the coefficients $\alpha, \beta, \gamma, \delta$, the barycentric coordinates of P for the tetrahedra T_1 and T_2. Designating the spheres circumscribing T_1 and T_2 by (ω_1, ρ_1), (ω_2, ρ_2), we then have

$$\rho_1^2 \sum \alpha = (\omega_1 P)^2 \sum \alpha + \sum (PA_1)^2 \alpha,$$
$$\rho_2^2 \sum \alpha = (\omega_2 P)^2 \sum \alpha + \sum (PA_2)^2 \alpha,$$

whence, as in the plane case, M being the midpoint of P_1P_2, M_a' its orthogonal projection on PA_3',

$$\sum (PA_1^2 - PA_2^2)\alpha = \sum (A_1A_2)(M_aP_a)\alpha = \sum (PA_3')(M_a'P)\alpha = 0,$$

and finally

$$(\omega_1 P)^2 - \rho_1^2 = (\omega_2 P)^2 - \rho_2^2,$$

which proves that P is the center of a fixed sphere orthogonal to the pedal sphere of P_1 and to those of all the points P_2 of Δ.

If we designate by Ω_A, Ω_B, Ω_C, Ω_D the trihedral angles $(P-B_3'C_3'D_3')$, $(P-C_3'D_3'A_3')$, $(P-D_3'A_3'B_3')$, $(P-A_3'B_3'C_3')$, whose edges are parallel to (B_1B_2, C_1C_2, D_1D_2), $\cdots$, and by $\theta_a, \theta_b, \theta_c, \theta_d$ the angles of like sense which Δ makes with the faces BCD, CDA, DAB, ABC of T, the barycentric coordinates $\alpha, \beta, \gamma, \delta$ of the center P of the sphere (P, σ) orthogonal to the spheres (ω_1, ρ_1), (ω_2, ρ_2) are equal to $(B_1B_2)(C_1C_2)(D_1D_2) \sec\theta_a \sin\Omega_A, \cdots$. Finally, for the square of the radius of the sphere (P, σ), we have

$$\sigma^2 = \sum \alpha\beta(A_1B_1)^2/(\sum \alpha)^2,$$

a formula which does not seem susceptible to transformation into a simple relation among the elements of the fundamental tetrahedron T.

---

* J. Neuberg, Mem. de la Soc. royale des Sciences de Liège, 1913, *loc. cit.*

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

Beginning with the August-September number of the MONTHLY the editor of this department will be Professor E. F. Beckenbach, University of California at Los Angeles, Los Angeles 24, California. Manuscripts should henceforth be sent to him.

ON THE EQUATION $\phi(x)=2m$

V. L. KLEE, JR., University of Virginia

Carmichael proved (*Bull. Amer. Math. Soc.*, 13 (1907), 241–243) that all solutions of the equation

$$\phi(x) = 2m \tag{1}$$

(where m is odd and >1 and ϕ is the Eulerian function) are of the form p^a, $2p^a$, where p is a prime of the form $4s-1$. Using this result we have at once the following

THEOREM. *The number of distinct solutions of* (1) *is exactly twice the number of ways in which m can be expressed in the form* (F) $n(2n+1)^k$, *where $2n+1$ is prime and k is an integer* $\geqq 0$.

For let $x=p^a=(4s-1)^a$ be a solution of (1). Then $\phi(p^a)=(4s-2)(4s-1)^{a-1}=2m$, whence, letting $n=2s-1$, we have $m=n(2n+1)^{a-1}$. Conversely, if $n(2n+1)^k=m$ is in the form (F), let $x_1=(2n+1)^{k+1}$ and $x_2=2x_1$. Then $\phi(x_1)=\phi(x_2)=2m$.

As corollaries we have:

I. *For* (1) *to have no solutions it is sufficient that m have no divisor $d>1$ for which $2d+1$ is prime.*

(Hence $\phi(x)$ is excluded from the value $2m$ for $m=7, 13, 17, 19, 31, \cdots$.)

II. *If $m=\Pi p_i^{a_i}$, where each prime p_i is of the form $3k_i+1$,* (1) *has no solutions.*

Since any product of numbers of the form $3k+1$ is itself of that form, any divisor of m must be of the form $3k+1$. Since $2(3k+1)+1$ is divisible by 3, the only expression of m in the form (F) is $1\cdot 3^0$, whence $m=1$, which contradicts the original assumption that $m>1$. (Hence $\phi(x)$ is excluded from the value $2m$ if m is any product formed from the factors 7, 13, 19, 31, $\cdots$.)

III. *If $m=p^2$, where p is a prime >3,* (1) *has no solution.*

We have $2p^2+1=2(p-1)(p+1)+3$, which is divisible by 3. (Since p must be of one of the forms $3k+1$ and $3k-1$.) Then if $2p+1$ is not prime, III follows from I. But if $2p+1$ is prime it is impossible to have $p^2=p(2p+1)^k$ unless $p=1$ and $k=0$. (Hence $\phi(x)$ is excluded from the value $2m$ for $m=25, 121, 289, \cdots$.)

IV. *The number of solutions of* (1) *is either* 0, 2, *or* 4.

Using the theorem, we need only show that m can be expressed in the form (F) in not more than two ways. Let $m=n_1(2n_1+1)^{k_1}=n_2(2n_2+1)^{k_2}$ be distinct expressions of m in the form (F). Clearly $(2n_1+1)^{k_1}$ divides either n_2 or $(2n_2+1)^{k_2}$. If it divides n_2 we have $n_2=q(2n_1+1)^{k_1}$, so $n_2>n_1$. But we must also have, $n_1=q(2n_2+1)^{k_2}$, whence $n_1>n_2$, which is a contradiction. Hence $(2n_1+1)^{k_1}$ divides $(2n_2+1)^{k_2}$. But then $n_1=n_2$. Hence for $k>0$, m has at most one expression in the form (F), and for $k=0$ the only possible expression is $m(2m+1)^0$.

THE SINE AND COSINE AS PROJECTION FACTORS

A. R. JERBERT, University of Washington

The following discussion of the sine and cosine as projections of a unit vector on the axes leads to a simple proof of the addition theorems for the sine and cosine. In figure 1 the terminal sides of *any* angle θ, and an equally general

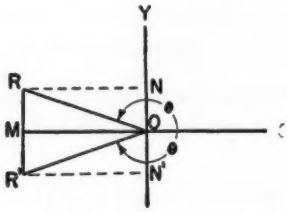

FIG. 1

angle $-\theta$, are evidently mirror images of each other in the x-axis. Therefore, OM is their common projection, so that, with $OR=OR'=1$,

$$\cos\theta = OM = \cos(-\theta). \tag{1}$$

Since MR and MR' extend in opposite directions from the "mirror," we have the equally evident, and equally general,

$$\sin(-\theta) = -\sin\theta. \tag{2}$$

The generality of the equations,

$$\begin{aligned} \sin(\theta+180) &= -\sin\theta \\ \cos(\theta+180) &= -\cos\theta, \end{aligned} \tag{3}$$

is also intuitive, since adding 180° to an angle reverses the terminal side and its projections on the two axes.

Cos θ is the *fundamental* projection ratio in the sense that it projects the terminal side of an angle θ, upon its initial side, which is the $+x$-axis if the angle is in "standard" position. Sin θ, on the other hand, projects the same unit vector on a line which is 90° *in advance*, namely the $+y$-axis. In figure 1, *e.g.*, sin $\theta=ON$ is the projection of OR on OY. Evidently, the same projection results from replacing $\theta=\angle XOR$, by $\theta-90=\angle YOR$ and employing the cosine in place of the sine. Thus,

$$\sin \theta = \cos (\theta - 90). \tag{4}$$

Since $(\theta - 90)$ is *any* angle, we may denote it by α, so that equation (4) becomes

$$\sin (\alpha + 90) = \cos \alpha. \tag{5}$$

Letting $\alpha = \beta + 90$, we have,

$$\sin (\beta + 180) = \cos (\beta + 90),$$

so that by reference to equation (3)

$$-\sin \beta = \cos (\beta + 90). \tag{6}$$

Cosine and sine projections which are directed oppositely to OX and OY are, nevertheless, identified with these directions, respectively, since the associated minus sign supplies the "correction" as in equations (3). With this understanding it can be stated, without exception, that $\cos \theta$ and $\sin \theta$ project the terminal side of any angle θ, upon its initial side, and a line 90° in advance of the initial side, respectively.

In figure 2 the angles α, β, and $(\alpha+\beta)$ are identified by drawing a curved arrow for β, and simply labeling the terminal sides of the other two, since they

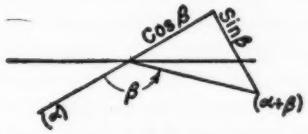

FIG. 2

are in "standard position." The functions of θ and $\theta \pm k360$ are identical so that there is no loss of *generality* in considering α and β as positive and less than 360°. The sine of $(\alpha+\beta)$ is the y-projection of its terminal side, 1. We may therefore compute this function by projecting the component vectors, $\cos \beta$ and $\sin \beta$, on the y-axis. Since the terminal side of α is initial for β, these vectors, in view of the remark following equation (6), are in the α and $\alpha+90$ directions; so that multiplication by $\sin \alpha$ and $\sin (\alpha+90) = \cos \alpha$, respectively, accomplishes the projection. Hence,

$$\sin (\alpha + \beta) = y\text{-proj. } (\cos \beta, \sin \beta) = \cos \beta \sin \alpha + \sin \beta \cos \alpha.$$

The remaining addition theorems may be obtained, in the usual manner, by taking advantage of equations (1) to (6).

Note by the Editor. Professor R. A. Johnson has called my attention to the fact that the proof of Feuerbach's theorem by H. F. Sandham, this MONTHLY, vol. 52, 1945, p. 571, is given in his *Modern Geometry*, p. 200, §321. The proof, he believes, is an old one, but he is unable to give a reference. R. Goormaghtigh also notes that this proof is given in practically the same form by W. J. Hodgetts in the *Mathematical Gazette*, 1944, p. 198. M. J. W.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

An Historical and Analytical Bibliography of the Literature of Cryptology. By J. S. Galland. Evanston, Northwestern University, 1945. 8+209 pages. $5.00.

The scope of this book is described very well in the following quotation from the preface:

"This volume is a register of the most important works that have been written, not only on the subject of cryptography, but also on its manifestations in related fields. Hence, the items presented will be seen to vary considerably in nature and content: some are scientific and technical treatises dealing with the application of cryptography to military, diplomatic, and commercial uses; others dwell on the more popular and more limited manifestations of this science, and appear in a variety of disparate forms,—in that of cuneiform, runic, and other kinds of hieroglyphic writings, in literary anagrams and acrostics, in symbolism of colors, gems, emblems, and insignia, in the jargon of children as well as that of thieves, in various types of universal languages and stenographic systems, in tramps' signs and semaphoric signals, and also in the crude devices and ingenious stratagems which were employed during primitive ages as a means of concealing and conveying thought and information."

The volume being reviewed consists of between 1600 and 1700 references to items on the above subjects. Some references are to short notes, while others are to works of several hundred pages. The references are listed alphabetically,—mostly by author's name, but occasionally by subject, or by the name of the company sponsoring the publication. Many of the listings are followed by comments concerning the referenced work. From these comments, which account for the "Analytical" in the title of the book, it is clear that the author devoted a great amount of work to the task of making the volume as useful as possible.

Since this review is written primarily for teachers of mathematics, it should be pointed out that, although the book contains a few references to mathematical articles having cryptology or cryptography in the title, those interested in the application of mathematics to cryptology may find that the volume is not as useful as might be desired from this point of view.

It seems to the reviewer that this book will be particularly useful to those interested in cryptology from the historical point of view, or as a recreation. The author's comments concerning the referenced works should be, in many cases, very helpful in estimating the probable usefulness of the item in question.

(Opinions and assertations expressed in this review are the private ones of the reviewer and are not to be construed as official or reflecting the views of the United States Navy Department or of the United States Naval Service at large.)

A. W. LANDERS, Lt. Comdr., USNR

Algebra. A Second Course. By R. O. Cornett. New York and London, McGraw-Hill Book Co., Inc., 1945. 13+313 pages. $2.00.

This text, which is planned for students who have had one year of high school algebra, could be used satisfactorily with students of this group who wish to study elementary mathematics in college for engineering or for later specialization in science or mathematics. It might serve as a text for those who are to study mathematics only as a part of their general education if the department prefers the more traditional course for this purpose.

Aside from the first two chapters there is no extreme departure from the traditional procedure or content. The intent of these two chapters to provide "an understanding of purpose and significance in advance of details" is certainly commendable. Only experience will determine if this can be done in so short a space. Chapter 1, Mental Gymnastics, contains numerous problems intended to assist the student in the development of what the author calls basic mental skills. Problems are planned to demonstrate visualization, manipulation of mental images, memorizing, systematic thinking, and concentration and endurance. In Chapter 2, the author compares a "systematic trial" method of solving problems involving two unknowns with the algebraic method. Uses of formulas in science and engineering are also brought out in this chapter.

Although the contents of chapters 3–10 are quite elementary, the book does contain sufficient material for a semester's work in algebra. There are 21 chapters in all. An indication of the extent of the material is given by chapter headings such as simple systems involving quadratics, systems leading to an equation of fourth degree, imaginary numbers, trigonometry, extraneous roots and irrational equations, progressions and the binomial theorem. The book contains the tables necessary for the topics covered. Answers to odd-numbered exercises are given. No use of determinants is made in the text.

There are numerous innovations which are worthy of mention. For example, students are led to solve stated problems involving two unknowns first by use of two variables, later being shown the possibility of using only one variable as a short cut. The space, through three chapters, given to quadratic equations in two variables in which pairs of equations are solved graphically and algebraically should give assistance in later work in analytic geometry and the calculus. In the chapter on the algebraic solution, frequent reference is made to the graphs of the preceding chapter.

Exercises on the use of graphs in the solution of stated problems are found in the chapter on Graphical Methods. Listed as optional topics in this chapter are the determination of the maximum or minimum value of a function from the graph and functional notation.

The organization of topics related to roots and radicals is different in that there is a chapter on square roots and radicals preceding quadratic equations and a later chapter on exponents and roots preceding logarithms. More complete than that usually found in freshman texts is the explanation of interpolation errors. In the chapter on logarithms, the exponential form is used even for rather difficult computations before the logarithmic notation is introduced.

Special techniques employed which are helpful include use of double asterisks (**) at the end of sentences to direct the student to stop reading and think carefully through what has just been discussed; classification of stated problems; use of the symbol $\doteq$ for approximate relationships; and review questions at the ends of chapters. The setting off of rules and of special formulas in boxes is desirable but is perhaps overdone in the more elementary parts of the text. The use of boys' names in the discussions on pages 84 and 254 seems a little juvenile for a college text.

The reviewer is most critical of what seems to him to be a carelessness in certain statements or expressions which might lead to misunderstanding on the part of the student even though later discussions of the same topics are faultless. This criticism is especially applicable to the discussion of roots of quadratic equations at the top of page 90 and to some of the later references to imaginary numbers as on page 154 and 187. Other examples are found in the paragraph on equations involving fractions on page 125 and the reference to the recording of an angle measurement on page 242.

The cover and whole make-up of the book are attractive. The pages are not crowded and are very readable. There are a number of misprints in the text, but none of a serious nature were noted. All things considered this book is a worthwhile addition to the present supply of algebra texts. Its strength lies in its fulfillment of a new need in the college field, through its elementary approach and its special emphasis on motivation. The practical selection of topics and carefully chosen exercises and applications are factors enhancing its value.

J. R. MAYOR

Elements of Calculus. By W. A. Granville, P. F. Smith, and W. R. Longley. Boston, Ginn and Co., 1946. 11+549 pages. $3.75.

The book bearing the above title is a revision of *Elements of the Differential and Integral Calculus* by the same authors. The only essential way in which it differs from the latter book is in the order of arrangement of the contents.

In the revised edition chapters dealing with indefinite and definite integrals, together with integration as a summation process and the Fundamental Theorem of the Integral Calculus, precede the differentiation of transcendental functions. This arrangement makes the revised edition particularly adaptable to courses in which the early introduction of both differentiation and integration is a desideratum.

H. P. EVANS

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Department of Mathematics, Oregon State College, Corvallis, Oregon.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 726. *Proposed by W. Nicholson, Chicago, Illinois*

If p and N are positive integers, $p>1$, show that N^p is the sum of N consecutive odd integers.

E 727. *Proposed by Frederick Mosteller, Princeton, N. J.*

What is the probability that a King and a Jack will appear side by side in a shuffled pack?

E 728. *Proposed by Victor Thébault, Tennie, Sarthe, France*

With the vertices A, B, C of an equilateral triangle as centers draw the circles (A), (B), (C) which are concurrent at the center O of the triangle, and then draw an arbitrary circle (D) passing through O. Show that the length of one of the common tangents to the circles (A) and (D), (B) and (D), (C) and (D) is equal to the sum of the lengths of the other two.

E 729. *Proposed by F. J. Duarte, Caracas, Venezuela*

Let x, y, z be three real positive numbers such that $x^3+y^3=z^3$, and set

$$\lambda = \sqrt[3]{3(x+y)}/(\sqrt[3]{z-x}+\sqrt[3]{z-y}).$$

Show that

$$(\sqrt[3]{2}+\sqrt[3]{4})/2 < \lambda < \sqrt[3]{3}.$$

E 730. *Proposed by J. H. Butchart, Arizona State College*

The interior angle bisectors of a triangle meet the noncorresponding sides of the medial triangle in six points which lie in pairs on the lines joining the points of tangency of the inscribed circle.

SOLUTIONS

A Ball in a Vase

E 687 [1945, 578]. *Corrected. Proposed by Victor Thébault, Tennie, Sarthe, France*

A heavy ball is gently dropped into a vase full of water, in the shape of a segment of a paraboloid of revolution. The size of the vase is given; that of the ball is such as to cause the maximum displacement. Find the radius of the ball.

Solution by E. P. Starke, Rutgers University. Let an axial cross section of the parabolic vase have the equation $x^2=2py$, and let its depth be a. Since the curvature of the parabola decreases monotonically as the distance from the vertex increases, the larger the ball the higher it will come to rest in the vase. Further, if a ball rests on the rim of the vase, not tangent to the parabola, a slightly smaller ball will sink farther in and displace more water. Hence, to determine a maximum, we need consider only balls which are tangent to the parabola and tangent to or intersecting the water surface.

If the ball is tangent to the parabola at the point (x_1, y_1), $y_1 \leqq a$, its center is at $(0, y_1+p)$ because of the subnormal property of the parabola. Suppose r is the radius of the ball and h is the depth of the submerged part. We then have

$$a = y_1 + p - r + h, \qquad r^2 = x_1^2 + p^2 = 2py_1 + p^2. \tag{1}$$

Now the volume of a spherical segment of radius r and height h is

$$V = \pi h^2(3r - h)/3. \tag{2}$$

Noting from (1) that

$$dy_1/dr = r/p, \qquad dh/dr = 1 - r/p,$$

we obtain from (2), by differentiation and simplification,

$$dV/dr = \pi hr(h + 2p - 2r)/p.$$

The usual tests for a maximum are satisfied when $2r=2p+h$, whence *the desired ball extends to a height* $2p$ *(the latus rectum of the generating parabola) above the surface of the water.* The value of r in terms of the given dimensions of the vase is found by eliminating h and y_1; it is

$$r = (2ap + 4p^2)^{1/2} - p.$$

Also solved by W. B. Campbell, M. P. de Regt, Roy Dubisch, R. E. Gaines, S. T. Parker, J. V. Pennington, Nathan Schwid, Elijah Swift, P. D. Thomas, and W. B. Zacharias.

Heronian Triangles with Sides in Arithmetic Progression

E 695 [1945, 516]. *Proposed by H. L. Lee, University of Tennessee*

Find triangles whose sides are integers in arithmetic progression, and whose areas are integers.

I. *Solution by D. W. Alling, Rochester, N. Y.* Let the sides of a triangle be the integers a, $(a+b)/2$, b, $a<b$, and where naturally a and b are both even or both odd and $b<3a$. Designate the area of the triangle by K. Then

$$K = \frac{a+b}{16}\sqrt{3(3a-b)(3b-a)} = \frac{a+b}{16}\sqrt{48ab-9(a+b)^2},$$

whence $16ab/3-(a+b)^2$ must be a squared integer and $3\mid ab$. Also $4\mid(a+b)$, because, together with $3\mid ab$, the assumption that $4\nmid(a+b)$ implies that $16ab/3\equiv(a+b)^2 \pmod{64}$, from which follows the contradiction that $(a+b)^2\equiv 0 \pmod{16}$. Thus $16\mid(3a-b)(3b-a)$ and $4\mid(3a-b)$, $4\mid(3b-a)$. Therefore the transformation

$$m = (3a-b)/4, \qquad n = (3b-a)/4, \qquad m<n,$$

may be made, m and n being both even or both odd, and the solutions of the diophantine equation

$$mn = 3x^2, \qquad x = 1, 2, \cdots$$

fulfill the conditions of the problem.

Following are some solutions:

x	m	n	triangle	K
1	1	3	3, 4, 5	6
2	2	6	6, 8, 10	24
3	1	27	15, 28, 41	126
	3	9	9, 12, 15	54
4	2	24	15, 26, 37	256
	4	12	12, 16, 20	96
	6	8	13, 14, 15	84

II. *Solution by E. P. Starke, Rutgers University.* Let a, $b=a-d$, $c=a+d$ be the sides. Then the area is $K=\frac{1}{4}a[3(a^2-4d^2)]^{1/2}$. If K is an integer, a^2-4d^2 must be three times a square, so that we have

$$3x^2 = a^2-4d^2, \qquad x = 4K/3a. \tag{1}$$

Now let the rational number $(a+2d)/3x$ equal m/n, where m, n are relatively prime integers. Then

$$n(a+2d) = 3mx, \qquad m(a-2d) = nx,$$

from which, by easy manipulation, we have

$$a:2d:x = (3m^2+n^2):(3m^2-n^2):2mn.$$

Hence the desired triangles are given by the formulae

$$a = 2k(3m^2+n^2), \quad b = 3k(m^2+n^2), \quad c = k(9m^2+n^2), \quad K = 6k^2mn(3m^2+n^2).$$

In order that a, b, c have no common factor, we choose $k=1/2$ when m, n are

both odd; $k=1/3$ when n is a multiple of 3; $k=1/6$ when both conditions hold; $k=1$ in all other cases.

An interesting special case arises if the sides are to be consecutive integers. Then $d=1$ and (1) becomes

$$a^2 - 3x^2 = 4.$$

The left member cannot be divisible by 4 except when a, x are both even: say $a=2A$, $x=2X$. If A_n, X_n is a solution of $A^2-3X^2=1$, so also is A_{n+1}, X_{n+1}, where

$$A_{n+1} = 2A_n + 3X_n, \qquad X_{n+1} = A_n + 2X_n. \tag{2}$$

From $A_0=1$, $X_0=0$ all solutions can be found by successive applications of (2). The first few solutions yield $a=4$, 14, 52, 194.

Also solved by F. A. Alfieri, Murray Barbour, P. T. Bateman, J. H. Cross, Monte Dernham, Daniel Finkel, B. A. Hausmann, W. L. Johnson, J. B. Kelly, M. Kirk, W. G. McGavock, Eric Michalup, C. D. Olds, S. T. Parker, P. A. Piza, L. S. Shively, R. H. Urbano, and the proposer. A number of these solutions gave only partial results and some of the solutions arrived too late to be considered for publication.

C. D. Olds referred to vol. II of Dickson's *History of the Theory of Numbers* for further problems of this type. Eric Michalup recorded 513 primitive solutions. L. S. Shively directed attention to the special solutions (939, 1876, 2813) and (193, 194, 195). The first is interesting because the triangle is so slender, the largest angle being approximately 175° 25′. The second is interesting because the triangle is so nearly equilateral. Lest someone may think that there is not a more nearly equilateral triangle, Shively also mentioned the triangle (805307497, 805308626, 805309755) with area 280818256141860864, inradius 232472576, and angles (to the nearest tenth of a second) 59° 59′ 59.5″, 60° 00′ 00.0″, 60° 00′ 00.5″.

Approximate Construction of Regular Pentagon

E 697 [1945, 578]. *Proposed by C. A. Murray, West Texas State College*

A certain geometry text raises the question whether the following procedure will inscribe a regular n-gon in a circle: AB being a diameter of the circle, construct an equilateral triangle ABC. Divide AB into n parts and let D be the second point of division from A. Draw CD, producing it to cut the circle at E. Is AE the side of a regular n-gon inscribed in the circle? For n equal to 3 or 4 the answer is readily affirmative. Does the procedure yield a regular pentagon for n equal to 5? If not, give a measure of the error.

I. *Solution by J. H. Cross, Lubbock, Texas.* The procedure does not give a regular pentagon; the arc so obtained is short by 2′ 48″. To see this let O be the center of the circles and r its radius. Draw EO. Evidently $AC=2r$, $AD=4r/5$, $DO=r/5$, $EO=r$, and $\sphericalangle DAC=60°$. From the law of tangents $\sphericalangle ADC=96°$ 35′ 13″ $=\sphericalangle EDO$. From the law of sines $\sphericalangle DEO=11°$ 27′ 35″. Hence $\sphericalangle AOE=71°$ 57′ 12″ instead of the required 72°.

II. *Solution by E. D. Schell, Arlington, Va.* Let the circle be of unit radius with its center at the origin. For $n=5$, the coordinates of C and D are $(0, \sqrt{3})$ $(-1/5, 0)$. The line CD has the equation $5\sqrt{3}x-y+\sqrt{3}=0$ and intersects the circle $x^2+y^2=1$ in the point E with coordinates

$$\{(-15-\sqrt{73})/76,\ \sqrt{3}(1-5\sqrt{73})/76\}.$$

The point A has the coordinates $(-1, 0)$, and the distance AE is found to be $\sqrt{(61-\sqrt{73})/38}=1.17491$. The side of a pentagon in a circle of unit radius, however, is $\sqrt{(5-\sqrt{5})/2}=1.17558$. The estimate is close enough to be useful since the error is only 7 parts in 10,000.

III. *Solution by Elmer Latshaw, Philadelphia, Pa.* Denote by θ the central angle subtended by side AE of the inscribed n-gon. By algebra and trigonometry we find

$$\cos\theta = (n-4)[3n+(n^2+16n-32)^{1/2}]/4(n^2-2n+4).$$

Solving this equation for various values of n gives:

n	θ by formula	$360°/n$	error per side
3	120.000°	120.000°	0.000°
4	90.000°	90.000°	0.000°
5	71.953°	72.000°	−0.047°
6	60.000°	60.000°	0.000°
7	51.528°	51.429°	0.099°
8	45.186°	45.000°	0.186°
9	40.278°	40.000°	0.278°
10	36.356°	36.000°	0.356°
20	18.633°	18.000°	0.633°

The procedure gives an exact construction for $n=3$, 4, and 6. For $n=5$ the error is small and $-0.047°$ per side; the error is positive and increases with n when $n>6$.

Also solved by G. W. Grotts, S. T. Parker, P. A. Piza, W. A. Rees, and R. H. Urbano.

A Construction in the Complex Plane

E 698 [1945, 578]. *Proposed by J. M. Feld, Brooklyn College*

Let the complex numbers a and b represent two points on the Gauss plane. If $\rho=re^{i\alpha}$, (r and α real), show how one can construct the point corresponding to $z=(a+\rho b)/(1+\rho)$.

Solution by E. A. Hedberg, M. I. T. Radar School, Boston. Solving the given relation for $-\rho$ we have

$$(z-a)/(z-b) = -\rho = re^{i(\alpha+\pi)},$$

whence

$$|z-a|/|z-b|=r, \tag{1}$$

$$\arg(z-a)-\arg(z-b)=\alpha+\pi+2n\pi, \tag{2}$$

where n is an integer.

The locus represented by equation (1) is a circle with its center on the line ab and dividing the segment from a to b internally and externally in the ratio r. The locus of equation (2) is an arc of a circle constructed upon segment ab as chord and tangent to the half line aR, where the positive angle baR is equal to α. Equation (2) is satisfied for any point on the arc lying on the same side of ab as aR. Therefore the intersection of this arc with circle (1) is the required point z.

If ρ is real, so that $\alpha=n\pi$, then z divides the segment ab internally or externally in the ratio r.

Also solved by G. W. Grotts, H. A. Luther (three ways), S. T. Parker and the proposer.

A Property of Some Cevians

E 699 [1945, 578]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3 be the feet of the altitudes, the symmedians, and the cevians through the circumcenter, on the sides BC, CA, AB of a triangle ABC. (a) The lines B_1C_1, B_2C_2, B_3C_3 are concurrent in a point M_1; C_1A_1, C_2A_2, C_3A_3 in a point M_2; A_1B_1, A_2B_2, A_3B_3 in a point M_3. (b) Triangle $M_1M_2M_3$ is homological to, and circumscribes, triangle ABC.

Solution by H. E. Fettis, Dayton, Ohio. Let H, K, O be respectively the orthocenter, symmedian point, and circumcenter of triangle ABC. Since the isogonal conjugates of these points are on a line, the points themselves are on a conic through the vertices of the triangle. Therefore $B(HKOA)=C(HKOA)$, and since the ranges $(B_1B_2B_3A)$ and $(C_1C_2C_3A)$ are sections of these pencils, $(B_1B_2B_3A)=(C_1C_2C_3A)$, whence B_1C_1, B_2C_2, B_3C_3 are concurrent at a point M_1. Similarly, C_1A_1, C_2A_2, C_3A_3 are concurrent at M_2; A_1B_1, A_2B_2, A_3B_3 are concurrent at M_3.

Now, let BM_1 intersect AC in N_2, and let BM_3 intersect AC in N_2'. We have

$$(N_2'B_1B_2B_3)=(BA_1A_2A_3)=A(BHKO)=C(BHKO),$$

and

$$(N_2B_1B_2B_3)=(BC_1C_2C_3)=C(BHKO),$$

so that $(N_2'B_1B_2B_3)=(N_2B_1B_2B_3)$, and N_2' coincides with N_2. Therefore B is on M_3M_1. Similarly, A is on M_2M_3, and C is on M_1M_2.

Further, let AM_1, BM_2 intersect in a point P, and let PM_3 cut BA, BC, AC in Q_3, R_3, R_3' respectively. Then

$$B_1(A_1C_1BA)=A(M_3M_1BN_2)=(M_3PQ_3R_3),$$

and

$$A_1(B_1C_1AB)=B(M_3M_2AN_1)=(M_3PQ_3R_3').$$

But $B_1(A_1C_1BA)=A_1(B_1C_1AB)$, since these are two harmonic pencils. Therefore R_3 and R_3' coincide at C and triangles ABC and $M_1M_2M_3$ are in perspectivity at P. The axis of perspectivity is $N_1N_2N_3$, the trilinear polar of P for triangle ABC.

Also solved by R. Bouvaist, J. H. Butchart, and L. M. Kelly.

Editorial Note. R. Bouvaist, employing barycentric coordinates, established the more general theorem: *Given a triangle ABC and three points D, E, F in its plane, whose cevians meet BC, CA, AB in D_a, D_b, D_c; E_a, E_b, E_c; F_a, F_b, F_c respectively. If the trilinear polars of D, E, F are concurrent, then*

$$D_bD_c,\ E_bE_c,\ F_bF_c \text{ are concurrent in } R,$$

$$D_cD_a,\ E_cE_a,\ F_cF_a \text{ are concurrent in } S,$$

$$D_aD_b,\ E_aE_b,\ F_aF_b \text{ are concurrent in } T.$$

Further, if two of the points D, E, F are isogonal conjugates for triangle ABC, then RST is homological to, and circumscribes, ABC.

By a slight improvement of Bouvaist's proof we may show that the restriction that two of the points D, E, F be isogonal conjugates is unnecessary, and that *RST is homological to, and circumscribes, ABC for any three points, D, E, F whose trilinear polars are concurrent.*

Following is an indication of the proof of the modified version of Bouvaist's generalization. If (x_D, y_D, z_D), (x_E, y_E, z_E), (x_F, y_F, z_F) are projective coordinates of D, E, F, the condition of the concurrency of their trilinear polars is

$$\begin{vmatrix} 1/x_D & 1/y_D & 1/z_D \\ 1/x_E & 1/y_E & 1/z_E \\ 1/x_F & 1/y_F & 1/z_F \end{vmatrix} = 0,$$

which may be written as

$$\begin{vmatrix} -1/x_D & 1/y_D & 1/z_D \\ -1/x_E & 1/y_E & 1/z_E \\ -1/x_F & 1/y_F & 1/z_F \end{vmatrix} = 0,$$

from which it follows that D_bD_c, E_bE_c, F_bF_c are concurrent in a point R. Similarly, D_cD_a, E_cE_a, F_cF_a are concurrent in a point S and D_aD_b, E_aE_b, F_aF_b are concurrent in a point T.

We readily obtain the following coordinates for R and S:

$$R:\ (1/y_Dz_E - 1/y_Ez_D,\ 1/x_Dz_E - 1/x_Ez_D,\ 1/y_Dx_E - 1/y_Ex_D),$$

$$S:\ (1/y_Dz_E - 1/y_Ez_D,\ 1/x_Dz_E - 1/x_Ez_D,\ 1/x_Dy_E - 1/x_Ey_D),$$

whence we see that R, S, C are collinear. Similarly, S, T, A and T, R, B are collinear, and RST circumscribes ABC.

Finally, RA and SB intersect in the point

$$P: \quad (1/y_D z_E - 1/y_E z_D,\ 1/z_D x_E - 1/z_E x_D,\ 1/x_D y_E - 1/x_E y_D).$$

Since the coordinates of P are symmetrical in those of D and E it follows that RST and ABC have P as center of perspectivity.

Needless to say, Fettis's proof also is capable of some generalization.

Kelly used trilinear coordinates. In terms of the notation of this note he first showed that *if, D, E, F are any three points in the plane of triangle ABC such that D_bD_c, E_bE_c, F_bF_c are concurrent in a point R, then D_cD_a, E_cE_a, F_cF_a are concurrent in a point S and D_aD_b, E_aE_b, F_aF_b are concurrent in a point T, and triangle RST is homological to, and circumscribes, triangle ABC.* He then completed the solution of the proposed problem by showing that the orthocenter, symmedian point, and circumcenter of ABC are three points like D, E, F.

Butchart established the first part of the theorem by cross-ratio methods. For the second part he referred to example 12, page 390 of Smith's *Conic Sections* (new edition, 1927).

Multinomial Coefficients

E 700 [1945, 579]. *Proposed by Arnold Dresden, Swarthmore College*

If h and i are respectively positive and non-negative integers, then

(a) $$\sum \frac{1}{p_0!p_1!\cdots p_i!} = \frac{(h+i-1)!}{i!(h-1)!h!},$$

where the sum on the left is extended over all sets of non-negative integers $p_0, p_1, \cdots, p_i$ which satisfy the conditions $p_0+p_1+\cdots+p_i=h$, $p_1+2p_2+\cdots+ip_i=i$;

(b) under the further restriction $i \geqq h-1$,

$$\sum \frac{1}{p_1!\cdots p_{i+1}!} = \frac{i!}{(i+1-h)!(h-1)!h!},$$

where the sum on the left is extended over all sets of non-negative integers $p_1, \cdots, p_{i+1}$ for which $p_1+\cdots+p_{i+1}=h$, $p_1+2p_2+\cdots+(i+1)p_{i+1}=i+1$.

Solution by D. W. Alling, Rochester, N. Y. In the identity

$$(1+x+\cdots+x^i)^h = (1-x^{i+1})^h(1-x)^{-h}$$
$$= (1-x^{i+1})^h \sum_{k=0}^{\infty} \binom{h+k-1}{k} x^k,$$

the coefficient of x^i on the right side is $\binom{h+i-1}{i}$, while by the multinomial theorem, the coefficient of x^i on the left side is $\sum h!/(p_0!p_1!\cdots p_i!)$, where the summation and the p's have the same meaning as defined in part (a) of the problem. Hence

$$\sum \frac{1}{p_0!p_1!\cdots p_i!} = \frac{(h+i-1)!}{i!(h-1)!h!}.$$

Similarly, comparing the coefficients of x^{i+1} in the identity

$$(x + x^2 + \cdots + x^{i+1})^h = x^h(1 - x^{i+1})^h(1 - x)^{-h}$$
$$= x^h(1 - x^{i+1})^h \sum_{k=0}^{\infty} \binom{h + k - 1}{k} x^k,$$

we have

$$\sum \frac{h!}{p_1!p_2! \cdots p_{i+1}!} = \binom{h + (i + 1 - h) - 1}{i + 1 - h} = \binom{i}{i + 1 - h},$$

which has meaning only if $h \leqq i+1$, and where the summation and the p's have the same meaning as defined in part (b) of the problem.

Also solved in the same way by H. S. Grant, M. S. Knebelman, E. D. Schell, F. Underwood, and the proposer. Schell pointed out that the necessary expansion theorems are given in Chrystal's *Algebra*, Vol. II, pp. 15 and 193.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4206. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider spheres with centers at the vertices of a tetrahedron $ABCD$ and radii equal respectively to k times the sum of the squares of the three opposite edges. Show that the sum of the squares of the distances from the four vertices to the center of the sphere orthogonal to the four spheres is equal to

$$[2(4k + 1)]^2R^2 - 2k(k + 1)\Sigma,$$

where R is the radius of the circumsphere and Σ means the sum of the squares of the six edges. Consider particular cases.

4207. *Proposed by K. W. Miller, Chicago, Ill.*

Consider the determinant $|a_{ij}| = \Delta$ of order n where

$$a_{ij} \neq 0, \quad \text{when} \quad (i - j)^2 \leqq k^2$$
$$a_{ij} = 0, \quad \text{when} \quad (i - j)^2 > k^2$$
$$1 \leqq i, j \leqq n$$
$$0 \leqq k \leqq (n - 1), \quad k \text{ arbitrary but fixed,}$$

that is, an nth order determinant Δ in which $(2k+1)$ consecutive diagonals symmetrical about and including the principal diagonal have *no* zero elements, while all other elements are zero in the two equal triangular corners of the determinant at each side of the central diagonal band. How many non-zero terms are there in the completely expanded determinant?

4208. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given an orthocentric tetrahedron, if two isogonal conjugate points are also conjugate with respect to the circumsphere, their pedal sphere is orthogonal to the sphere whose center is the complement of the orthocenter for the linear net determined by the spheres circumscribed and conjugate.

SOLUTIONS

De L'Hospital's Rule

4151 [1945, 163]. *Proposed by B. M. Stewart, Michigan State College*

Let O be a point at which a given curve has a second derivative; let the tangent and normal at O serve respectively as x, y axes, the equation of the curve becoming $y=f(x)$; and let P; (x, y) be a point on the curve, say with positive x. Denote the arc length OP by s and locate the point S: $(s, 0)$. The line SP intersects the y-axis in the point B:$(0, b)$. If R indicates $[1+(y')^2]^{3/2}/y''$, show that the limiting position of the point B as P approaches O is such that $\lim b = 3 \lim R$.

This is a generalization of a problem in the calculus of Granville, Smith, and Longley, 1934, p. 177, ex. 20.

Solution by L. A. Santaló, Rosario, Argentina. Let $x=x(s)$, $y=y(s)$ be the parametric equations of the curve $y=f(x)$ with the arc length s as parameter. If we denote by accents derivatives with respect to s, it is well known that $(x')^2+(y')^2=1$ and $x'x''+y'y''=0$. From this and from $y'((0)=0$ and $R=1/y''$, we deduce $x''(0)=0$. Hence

$$(1) \qquad x(0) = y(0) = 0; \quad x'(0) = 1, \; y'(0) = 0; \quad x''(0) = 0, \; y''(0) = 1/R_0;$$

where

$$R_0 = \lim_{s \to 0} R.$$

From the equation of the straight line SP we deduce $b=OB=sy/(s-x)$. Applying l'Hospital's rule twice, we have

$$\lim_{s \to 0} b = \lim_{s \to 0} \frac{sy}{s-x} = \lim_{s \to 0} \frac{y+sy'}{1-x'} = \lim_{s \to 0} \frac{2y'+sy''}{-x''}.$$

From the fact that $x'x''=-y'y''$, and from (1), we deduce that

$$\lim_{s \to 0} b = \lim_{s \to 0} \frac{2y'x'+sy''x'}{y'y''} = 2R_0 + \lim_{s \to 0} \frac{sx'}{y'},$$

and, if we apply this rule again, we find that

$$\lim_{s\to 0} b = 3R_0.$$

This problem may be generalized in the following way. Let us consider two plane curves C, C_1 tangent at O, and let R and R_1 be their radii of curvature at O. Let P be a point on C and P_1 a point on C_1 with the condition that arc length $OP=s$ equals arc length OP_1. The straight line of PP_1 meets the normal at O in point B; then we have

$$\lim_{s\to 0} OB = \frac{3}{\frac{1}{R}+\frac{1}{R_1}}. \tag{2}$$

If $R_1=\infty$ we obtain the proposed problem. If $R=R_1$, the formula (2) is not applicable. In this case, if $(dR/ds)_0 \neq (dR_i/ds)_0$, we find that

$$\lim_{s\to 0} OB = \tfrac{4}{3}R. \tag{3}$$

From (2) and (3) may be deduced a geometrical example referring to a function of two variables for which the order of limits cannot be interchanged. Let us consider the curves C and C_1 tangent at O with $R\neq R_1$, and suppose that we wish the limiting position of the point B as $R_1\to R$ and $s\to 0$. If we take first $s\to 0$ and then $R_1\to R$, by (2) it follows that $\lim OB=3R/2$. But if we take first $R_1\to R$ and then $s\to 0$, we have by (3) $\lim OB=4R/3$.

All these results and the proposed problem are treated in our paper *Algunas propriedades infinitesimales de las curvas planas*, Mathematicae Notae, Año I, pp. 128–144, 1941.

Solved also by Mrs. R. C. Buck, Howard Eves, J. F. Heyda, J. B. Kelly, A. Sisk, C. E. Springer, J. T. Webster, and the proposer.

Editorial Note. The solutions by Eves and Springer used formulas for the coordinates x, y of a point on a curve in terms of the arc length parameter s, the first referring to Graustein's *Differential Geometry*, p. 39, equations (46), and the second to Eisenhart's *An Introduction to Differential Geometry*, p. 26. The remaining solvers use three applications of de l'Hospital's theorem in a manner somewhat similar to the above solution.

We may also use vector methods as follows. Consider the curve as concave upward at O passing through P in its neighborhood, say to the right; and let $\mathbf{t}$ and $\mathbf{n}$ be the unit vector tangent and normal at O forming a right hand system. Let $\mathbf{r}$ be the vector of P, the origin of vectors being arbitrarily chosen, where $\mathbf{r}$ is a function of s, the arc length to P with the positive sense $\overrightarrow{OP}$. Then at O we find the values

$$\mathbf{t}=d\mathbf{r}/ds=\mathbf{r}';\ \mathbf{t}'=\mathbf{r}''=\kappa\mathbf{n},\ \kappa=1/R;\ \mathbf{n}'=-\kappa\mathbf{t};\ \mathbf{t}''=\kappa'\mathbf{n}-\kappa^2\mathbf{t};$$
$$\mathbf{t}'''=(\kappa''-\kappa^3)\mathbf{n}-3\kappa\kappa'\mathbf{t};\quad etc. \tag{1}$$

Using these results we have

$$\overrightarrow{OP} = d\mathbf{r} = \mathbf{t}ds + \frac{\kappa\mathbf{n}}{2!}(ds)^2 + \frac{(\kappa'\mathbf{n} - \kappa^2\mathbf{t})}{3!}(ds)^3 + \cdots. \tag{2}$$

Since $\overrightarrow{OS} = \mathbf{t}ds$, the sum of the remaining terms of (2) is $\overrightarrow{SP}$; and then $b\mathbf{n} = \mathbf{t}ds + \overrightarrow{ASP}$. This last equation gives

$$\begin{aligned} 1 &= \frac{A(ds)^2}{3!}\left[\kappa^2 + \tfrac{3}{4}\kappa\kappa' ds + \cdots\right]; \\ b &= \frac{A(ds)^2}{2}\left[\kappa + \frac{\kappa' ds}{3} + \cdots\right]; \\ b &= 3R\,\frac{\left[1 + \dfrac{R\kappa'}{3}ds + \cdots\right]}{\left[1 + \dfrac{3R\kappa'}{4}ds + \cdots\right]}. \end{aligned} \tag{3}$$

The limit value of b is now easily obtained.

Trigonometric Expansion of Binomial Coefficients

4152 [1945, 163]. *Proposed by William J. Taylor, Washington, D. C.*

Prove the following trigonometric expansion for the binomial coefficient

$$\frac{N!}{\left(\frac{N+x}{2}\right)!\left(\frac{N-x}{2}\right)!} = \frac{2^N}{N}\sum_{m=1}^{N}\left(\cos\frac{m\pi}{N}\right)^N \cos\frac{m\pi x}{N}, \qquad -N < x < N.$$

I. *Solution by J. B. Kelly, Hampton, Va.* Set $w = e^{\pi i/N}$, then the right member in the problem becomes

$$\begin{aligned} &\frac{1}{2N}\sum_{n=1}^{N}(w + w^{-n})^N(w^{nx} + w^{-nx}), \\ &\quad = \frac{1}{2N}\sum_{n=1}^{N}\sum_{k=0}^{N}\binom{N}{k}w^{n(N-2k)}(w^{nx} + w^{-nx}), \\ &\quad = \frac{1}{2N}\sum_{k=0}^{N}\binom{N}{k}\sum_{n=1}^{N}\left[w^{n(N-2k+x)} + w^{n(N-2k-x)}\right]. \end{aligned} \tag{1}$$

Since $-N < x < N$, we have

$$\sum_{n=1}^{N} w^{n(N-2k+x)} = 0, \tag{2}$$

unless $N - 2k + x = 0$, in which case the sum in (2) is N. Similarly,

$$(3) \qquad \sum_{n=1}^{N} w^{n(N-2k-x)} = 0, \qquad -N < x < N,$$

unless $k=(N-x)/2$, and the sum in (3) is again N. Hence (1) becomes

$$\frac{1}{2N}\left[N\binom{N}{\frac{N+x}{2}} + N\binom{N}{\frac{N-x}{2}}\right] = \frac{N!}{\left(\frac{N+x}{2}\right)!\left(\frac{N-x}{2}\right)!},$$

and the proof is complete.

II. *Solution by C. D. Olds, San Jose State College, Calif.* Let $w=e^{2\pi i/N}$, and use the binomial theorem to obtain in turn the expansions

$$(1+w)^N = \binom{N}{0} + \binom{N}{1}w + \binom{N}{2}w^2 + \cdots + \binom{N}{N}w^N,$$

$$(1+w^2)^N = \binom{N}{0} + \binom{N}{1}w^2 + \binom{N}{2}w^4 + \cdots + \binom{N}{N}w^{2N},$$

. . . . . .

$$(1+w^N)^N = \binom{N}{0} + \binom{N}{1}w^N + \binom{N}{2}w^{2N} + \cdots + \binom{N}{N}w^{N^2}.$$

Multiply the rows through respectively by $w^{-r}, w^{-2r}, \cdots, w^{-Nr}$, where r is an integer, $0<r<N$, and then add. We obtain at once

$$\sum_{m=1}^{N}(1+w^m)^N w^{-rm} = N\left[\binom{N}{r} + \binom{N}{N+r} + \cdots\right] = N\binom{N}{r},$$

since

$$\binom{N}{kN+r} = 0.$$

On the other hand

$$\sum_{m=1}^{N}(1+w^m)^N w^{-rm} = \sum_{m=0}^{N}(w^{m/2}+w^{-m/2})^N \omega^{m(N-2r)/2},$$

$$= \sum_{m=0}^{N}\left(2\cos\frac{m\pi}{N}\right)^N\left[\operatorname{cor}\frac{m(N-2r)\pi}{N} + i\sin\frac{m(N-2r)\pi}{N}\right]$$

$$= N\binom{N}{p}.$$

Equating real and imaginary parts we get

$$\binom{N}{r} = \frac{2^N}{N}\sum_{m=1}^{N}\left(\cos\frac{m\pi}{N}\right)^N \cos\frac{m(N-2r)\pi}{N}.$$

Letting $x=N-2r$ we see that this is the expression we set out to verify.

Remarks: The same method can be used to show that if $q<N$ and $0\leqq r\leqq q-1$, then

$$\binom{N}{r}+\binom{N}{q+r}+\binom{N}{2q+r}+\cdots=\frac{1}{q}\sum_{m=0}^{q-1}\left(2\cos\frac{m\pi}{q}\right)^N\cos\frac{m(N-2r)\pi}{q}.$$

This leads to the interesting special cases:

$$\binom{N}{0}+\binom{N}{3}+\binom{N}{6}+\cdots=\frac{1}{3}\left(2^N+2\cos\frac{N\pi}{3}\right),$$

$$\binom{N}{1}+\binom{N}{4}+\binom{N}{7}+\cdots=\frac{1}{3}\left(2^N+2\cos\frac{(N-2)\pi}{3}\right),$$

and so on. This expression was first obtained, I believe, by C. Ramus, *Journ. f. Math.* vol. 11 (1834), p. 353.

Solved also by the professor.

Editorial Note. The result in Old's remark may be obtained by Kelly's method. The integers r, q, N are such that $1\leqq q<N$, $0\leqq r\leqq q-1$. Set $w=e^{\pi i/q}$, then

$$\begin{aligned}\frac{1}{q}\sum_{m=0}^{q-1}&(2\cos m\pi/q)^N\cos m(N-2r)\pi/q\\ &=\frac{1}{2q}\sum_{m=0}^{q-1}(w^{-m}+w^m)^N\left[w^{m(N-2r)}+w^{-m(N-2r)}\right]\\ &=\frac{1}{2q}\sum_{m=0}^{q-1}\sum_{k=0}^{N}{}_NC_k\left[w^{2(N-r-k)m}+w^{2(r-k)m}\right]\\ &=\frac{1}{2q}\sum_{k=0}^{N}{}_NC_k\sum_{m=0}^{q-1}\left[w^{2(N-r-k)m}+w^{2(r-k)m}\right].\end{aligned}\tag{1}$$

The second sum consists of two geometric series of q terms each, and in first, if $N-r-k$ for the fixed k is neither zero nor a multiple of q, the sum of the series is zero. If on the contrary it is zero or a multiple q, the sum is q, and in this case k has the values $N-r$, $N-r-q$, $N-r-2q$, $\cdots$, $N-r-[(N-r)/q]q$. In a similar manner the second series is zero, if $r-k$ is neither zero nor a multiple of q; if it is, then the sum is again q, and $k=r$, $r+q$, $r+2q$, $\cdots$, $r+[(N-r)/q]q$. Hence

$$\sum_{t=0}^{[(N-r)/q]}{}_NC_{r+tq}=\frac{2^N}{q}\sum_{m=0}^{q-1}\left(\cos\frac{m\pi}{q}\right)^N\cos\left[\frac{m(N-2r)\pi}{q}\right].\tag{2}$$

Tetrahedron and Rolling Sphere

4153 [1945, 163]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A sphere (S), radius r, rolls on the plane of the face BCD of the tetrahedron

$ABCD$ so that its center S lies on a fixed sphere concentric with (O) the circumsphere of $ABCD$. Show that: (1) The tetrahedron $BCDS$ is inscribed in a fixed sphere (S'). (2) The sphere tangent interiorly to (S) and passing through the points B, C, D envelopes a fixed sphere concentric with (O). If OS' is prolonged by $S'A'=r/2$, the sphere with center A' and passing through B, C, D is orthogonal to (S).

Solution by Howard Eves, College of Puget Sound.

(1) As (S) rolls in the prescribed manner, the center S traces a circle σ on the sphere (O). This circle σ and the circle α, passing through B, C, D, have a common axis. Therefore σ and α are cospherical on a sphere (S').

(2) It is clear that all spheres passing through B, C, D and interiorly tangent to the various positions of (S) coincide, and the theorem of part (2) of the problem becomes trivial.

(3) Let OS' cut the plane BCD in point K. Denote $S'K$ by k, and let s be the radius of (S'), and t the radius of the sphere (A') having center A' and passing through B, C, D. Let L be the foot of the perpendicular from S on OS'. Then

$$\begin{aligned} t^2 &= A'K^2 + KD^2 = A'K^2 + (S'D^2 - S'K^2) \\ &= (s^2 - k^2) + (k + r/2)^2 = s^2 + r^2/4 + kr. \end{aligned}$$

Also

$$\begin{aligned} A'S^2 &= A'L^2 + SL^2 = A'L^2 + (S'S^2 - S'L^2) \\ &= (3r/2 + k)^2 + [s^2 - (r + k)^2] \\ &= s^2 + 5r/4 + kr. \end{aligned}$$

It follows that

$$A'S^2 = t^2 + r^2,$$

whence (A') and (S) are orthogonal.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

A conference on Algebra will be held at the University of Chicago during the week of July 15–19, 1946. A program is being planned consisting of approximately twelve invited addresses. Several of the sessions will emphasize recent contacts of algebra with other fields such as topology, geometry and function theory. There will also be a session devoted to pure algebra.

The Society for the Promotion of Engineering Education has instituted a George Westinghouse Award in Engineering Education to be given to the college or university teacher adjudged to have contributed most to the successful teaching of engineering students. The amount of the award is $1,000, to be conferred annually. For details inquire of Dr. Harry S. Rogers, Polytechnic Institute of Brooklyn.

Dr. H. W. Alexander has been appointed to a professorship at Adrian College, Adrian, Michigan.

Professor L. D. Ames of the University of Southern California has retired.

Professor H. F. Bohenblust of Indiana University has been appointed to a professorship at California Institute of Technology.

Professors E. T. Browne and M. A. Hill and Associate Professor E. A. Cameron have returned to the University of North Carolina.

Assistant Professor J. H. Curtiss of Cornell University has been appointed assistant to the director of the National Bureau of Standards in Washington, D. C.

Assistant Professor W. J. Dixon of the University of Oklahoma has been promoted to an associate professorship.

Dr. D. C. Duncan of Los Angeles City College has been appointed chairman of the mathematics department at East Los Angeles Junior College.

Associate Professor Samuel Eilenberg of the University of Michigan has been appointed to a professorship at Indiana University.

Captain H. T. Engstrom, U.S.N.R. has been awarded the rank of Officer in the Military Division of the Order of the British Empire.

Dr. Benjamin Epstein has been appointed mathematician in the *Coal* Research Laboratory of Carnegie Institute of Technology—not in the *Cost* Research Laboratory as was previously announced.

Dr. R. W. Erickson has been appointed to an associate professorship at Carleton College, Northfield, Minnesota.

Professor V. G. Grove and Assistant Professors A. W. Jones, C. P. Wells and J. W. Zimmer have returned to Michigan State College from leaves of absence.

Assistant Professor S. G. Hacker has been promoted to an associate professorship at Washington State College.

J. F. Hubbard, formerly of the W. B. Rogers School, Hyde Park, Boston, is now teaching in the School for Veterans in the High School of Commerce Building, Boston.

Professor H. K. Justice of the University of Cincinnati has been appointed assistant dean and director of admissions in the College of Engineering.

Assistant Professor Mark Kac of Cornell University and Associate Professor J. W. T. Youngs of Indiana University have been awarded Guggenheim Fellowships.

Dr. E. S. Kennedy of the University of Alabama has been appointed to an adjunct professorship at the American University of Beirut, Beirut, Lebanon.

Dr. P. A. Lagerstrom of Bell Aircraft Corporation has been appointed research aerodynamicist with the Douglas Aircraft Company of Santa Monica, California.

Dr. Walter Leighton of the Rice Institute has been appointed to a professorship at Washington University.

Professor W. T. Martin of Syracuse University has been appointed to a professorship at Massachusetts Institute of Technology.

Associate Professor G. M. Merriman of the University of Cincinnati has been promoted to a professorship.

Assistant Professors Ivan Niven and A. H. Smith of Purdue University have been promoted to associate professorships.

P. B. Norman has been appointed to an assistant professorship at the School of Engineering of Pratt Institute.

Dr. M. J. Norris has been appointed to an assistant professorship at the College of St. Thomas, St. Paul, Minnesota.

Associate Professor V. C. Poor of the University of Michigan has retired with the title of professor emeritus.

Drs. Maxwell Reade and G. S. Young of Purdue University have been promoted to assistant professorships.

Assistant Professor Henry Scheffé of Syracuse University has been appointed to an associate professorship in engineering at the University of California at Los Angeles.

Dr. R. W. Shephard of Bell Aircraft Corporation has been appointed to an assistant professorship at Purdue University.

W. F. Smith has been appointed to an assistant professorship at the University of Detroit.

Professor J. L. Synge of Ohio State University has been appointed head of the applied mathematics division of Carnegie Institute of Technology. Professor Tibor Radó has been appointed chairman of the department of mathematics at Ohio State University.

Assistant Professor F. J. Taylor of the College of St. Thomas has been promoted to an associate professorship.

Associate Professor R. M. Trimble of the University of North Carolina has been promoted to a professorship.

Drs. R. W. Wagner and E. P. Vance of Oberlin College have been promoted to assistant professorships.

Associate Professor D. L. Webb of Texas Technological College has been appointed to an assistant professorship at the University of Arizona.

Lieutenant Commander R. H. Wilson, Jr. of the United States Naval Academy has been appointed to an assistant professorship at Temple University.

Dr. P. M. Young has returned to Miami University as an assistant professor.

The following appointments to instructorships are announced:
Agricultural and Mechanical College of Texas: Dr. R. E. Basye
College of St. Thomas: Eric Clamons, Donald Lewis, Dr. Takashi Terami
Lehigh University: S. D. Bernardi, Dr. H. K. Brown
Missouri School of Mines: Neil Lockwood
North Carolina State College: S. R. Baker
Oregon State College: Dr. J. C. R. Li
Pratt Institute (School of Engineering): G. C. Helme
Purdue University: Dr. Bernard Dimsdale, H. L. Harter, J. P. Wayne, Dr. John Dyer-Bennet, G. L. Kinnett, Dr. Sam Perlis, G. C. Sipple
Rutgers University: Dr. Lee Byrne
Syracuse University (Endicott Branch): Frances M. Wright
Tulane University: E. P. Miles, Jr.
The University of Buffalo: H. W. Baeumler, Ruth A. Brendel, Irene L. Fisher, Mrs. Magdalene M. Hettler, C. E. Maley, Audrey L. Strabel, Mrs. Rosemary R. Trautman
University of Michigan: Dr. Samuel Kaplan
University of Missouri: L. M. Kelly
University of Pennsylvania: Dr. Y. C. Wong
The University of Rochester: Norman Gunderson
University of Southern California: Dr. P. J. Kelly

Dr. Fletcher Durell died March 25, 1946. He was a charter member of the Association.

Professor Laurence Hadley of Purdue University died March 21, 1946. He was a charter member of the Association.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE GRADUATE RECORD EXAMINATION

The Graduate Record Examination is a project begun in 1937 under the sponsorship of the Graduate Schools of Harvard, Yale, Princeton, and Columbia Universities in collaboration with the Carnegie Foundation for the Advancement of Teaching. Since that time the growth of the project has been so rapid that the Graduate Record Examination has become an important, and a controversial, feature of American academic life. The following paragraphs have been selected from reports published by the Graduate Record Office and by the Carnegie Foundation for the Advancement of Teaching.

"During a period of confusion in the calculation of time credits and course credentials on which progress in American colleges and universities has traditionally rested, this Examination is believed to offer an element of stability theoretically sounder than the customary measures when these were used alone and certainly deserving of thoroughgoing trial in the present emergency. Broadly covering the student's total current academic resources, the results provide a series of comparable measures that make his available equipment from whatever source immediately intelligible at any institution.

"Early in 1944, foreseeing the problem of the returning veteran provided with partial or irregular credentials of the old type, a group of graduate schools familiar with the tests (Harvard, Yale, Michigan, Wisconsin) urged that they be made generally available. The proposal finally issued by them to their colleagues in other schools was to the effect that in return for the widespread requirement or recommendation of the Examination to applicants, the Carnegie Foundation would undertake to make it readily accessible throughout the country at a small fee, reimbursing a part of the cost.

"The suggestion that the Examination could be conveniently taken and the results submitted with the student's application before admission appealed strongly to the graduate schools at large, especially to those receiving students from a distance or from little-known institutions. By June, 1944, the cooperating graduate schools numbered 57, all of which either required or invited the applicant for admission to submit results of the Examination as auxiliary credentials. During the year now closing (June, 1945) this number has increased to 98, of which 38 make the Examination a requirement either of all or of certain classes of its applicants.

"In addition to the 98 sponsoring institutions there are (June 30, 1945) 110 other colleges or universities at which official examiners under contract with

the Foundation are available to give the tests. Of these 208 institutional centers, 11 are in Canada.

"Although the Examination originated from the need of graduate schools of liberal arts and sciences for a trustworthy picture of the student's underlying education, it is obvious to those familiar with it that its usefulness is much wider. Any professional institution of higher than freshman level—medicine, law, business, engineering—that is concerned about the intellectual attainment and emphases of its prospective students can get accurate and comprehensive information from these tests. Likewise in most cases norms will be found indicating the performance of large numbers of students having in view the particular professional goal desired.

"All told, during the past year, 1944–45, 6,535 students took the Graduate Record Examination. Of these, 1,213 were individual applicants tested at over 200 different centers—many of them at the central office in New York—while 5,322 were in 101 institutional groups tested for the routine purposes of the institution concerned. These bring to a total of more than 45,000 the number of complete scorings of the G.R.E. that have been made since its inception.

"The Graduate Record Examination is administered in two parts. The general or elementary part covers seven subject matter fields and a general vocabulary test. This part of the Examination is taken by all students without regard to their major subject. Separate scores are reported for each subject as follows: Mathematics, Physics, Chemistry, Biology, Social Studies, Literature, Fine Arts, and Verbal Factor. These scores which constitute a record of general education may be graphically shown in the Report Chart, which is made available to the student.

"In addition to the general tests, the record will include scores on a 90-minute specialized examination suitable for testing students in their subjects of major study. At the time of registering for the Examination, the student indicates his choice of Advanced Test. Such tests are offered in History of Fine Arts, Biological Science, Chemistry, German, Economics, French, Geology, History, Government, Psychology, Literature, Mathematics, Physics, Engineering, Sociology, and Philosophy. Each student elects one advanced subject test.

"The questions employed in the tests which comprise the Graduate Record Examination require answers that are unequivocal and admit of precise tabulation. Most of the questions are written in the familiar multiple-choice form; some require the student to complete a statement.

"The Examination is administered as a unit in two half-day sessions. The sessions are scheduled for separate days to minimize fatigue. The actual working time is about three hours per session; the formal schedule requires a morning or afternoon of four hours to allow for a mid-session intermission and additional time for students who may require more than normal time. The separate tests are given within specified time limits. These time limits are generous, and most students will have finished as much of the test as their knowledge permits before the time limit is reached.

"Primary emphasis in the General Mathematics Test is placed on skill in quantitative thinking. The problems presented in the test vary widely in their derivation but the emphasis is generally of a practical nature. The test includes problems dealing with the interpretation of data in graphical and tabular form, with inconsistencies in numerical data, with the transformation of verbal descriptions into mathematical descriptions and the reverse transformations, with evaluation of the soundness of conclusions drawn from numerical data, with the recognition of assumptions that must be made before valid conclusions can be drawn, and with similar accepted outcomes for the general development of college students. The items in this test were selected from a variety of situations which the advanced college student is likely to meet during his college study. Quantitative problems from various fields of study are included."

Inquiries regarding the Examination may be addressed to the Secretary of the Graduate Record Examination, 437 West 59th Street, New York 19, N. Y.

THE VETERAN EDUCATION PROBLEM

The number of veterans desiring to take advantage of the educational provisions of Public Laws 346 and 16 has far exceeded all of the early estimates. By February 28, a total of almost 1,500,000 applicants had been received by the Veterans Administration; the trend is indicated by the fact that 340,000 of these applications were received in January and 456,000 in February. Not all of these "cases in file" are now in education and training; some have completed or discontinued their training, and many have applied to receive their certificate of eligibility and entitlement but have not yet enrolled in an educational institution. Dr. Francis J. Brown, consultant for the American Council on Education, recently stated that "four million veterans will take advantage of educational benefits in five years—if the benefits are available."

On the basis of the number of applicants, a previous estimate of 750,000 veterans enrolled in our colleges and universities by September, 1946, appears to be conservative. So, too, does the estimate that the total enrollment in higher education by next fall will be at least 25% above the previous peak enrollment of 1,480,000 in 1939–40. The total number will also be increased by at least 100,000 high school graduates if Selective Service is modified on May 15 to include only men over 20 years of age. Never have the colleges faced so serious a responsibility nor so great an opportunity! U. S. Solicitor General J. H. McGrath analyzed the situation now facing colleges and universities in the following words: "The great bottleneck now is housing, and it undoubtedly will continue to vex us; the next problem will be where to find instructors even if we do find more housing."

The editor of this section recently made a survey of a large number of mathematics departments to ascertain plans and policies for taking care of the greatly increased enrollment in mathematics courses. It appears up to the present time that most departments have succeeded in obtaining sufficient staff members to handle the large teaching load; frequently, however, it has been necessary to

increase slightly the credit-hour load of each instructor and the class size. In general, the maximum class size seems to be 30 or 35. Regular staffs are being augmented in many cases by the return to the class room of married women who formerly taught mathematics. Some of the better prepared high school teachers are being drawn into the college field, and more graduate students are being given teaching assignments. Since a large part of the heavy load at the present time is to be found upon the elementary level, such solutions of the teaching problem seem to be working out very well, although, as one person points out, "the general lowering of the academic level of teachers does not do justice to the minority of superior students who would profit by contact with teachers of higher academic qualifications." Several institutions are developing plans for the closer supervision of inexperienced instructors.

Some institutions are making extensive and reasonably effective use of guidance and placement examinations in the registration of students. In spite of all precautions, however, many institutions find during the early weeks of a semester that a large number of students are improperly registered, so there is a considerable amount of re-registration after a term starts. To accommodate those veterans who are unprepared to handle college work, a large number of review courses are being conducted. These courses may take up high school algebra, plane and solid geometry, and even arithmetic. At least one institution reports the operation of special tutoring sessions at which veterans may obtain help.

To take care of the shortage in class rooms, several institutions have extended the length of the teaching day. Late afternoon and night classes seem to be common. In some institutions the noon hour is being used.

Advanced undergraduate students are generally being used to advantage. For example, many senior students are being assigned as assistants to instructors having large classes. One correspondent reported that he knew of some mathematics majors at the senior level who taught classes in freshman mathematics at the conclusion of the first World War. He indicated his belief that a good student at the senior level would derive much benefit from teaching one freshman course, and "he would probably do a good job of it."

One large midwestern institution is operating seven large classes in mathematics during the present term in addition to many sections of the usual size. Three of the large classes are five-hour courses in college algebra and trigonometry, and four of them are five-hour courses in analytic geometry; each large class contains 100 to 125 students. On four days of the week, the instructor of a large class meets the group as a unit in a lecture period; on the fifth day, each class is divided into four sections, all under the supervision of the instructor who has given the lectures. The problem of scheduling the four sections on the fifth day was easily solved by analyzing the vacant periods of the class members on that day. An instructor handling one of the large classes has his teaching load figured at 12 hours for one group. Also, a student assistant is assigned to each large class.

Virtually all institutions are seeking additional staff members for the fall term, since most colleges and universities are assuming that there will be at least a 25% increase in enrollment.

THE LOUISIANA-MISSISSIPPI EDUCATIONAL COMMITTEE

W. V. Parker and H. T. Karnes, Louisiana State University

In 1941, the Mathematics Section of the Louisiana College Conference, the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics, and the Louisiana-Mississippi Section of the Mathematical Association of America created a standing Joint Committee, composed of two members from each of the organizations. This Committee was charged with the study of the curricula of high schools and colleges, and was asked to make recommendations to the proper authorities, from time to time, concerning mathematical requirements for graduation and the training of teachers of secondary mathematics.

The Joint Committee was very active through the year 1942. During the war years little work was done. However, the Committee was reactivated at the recent Ruston meeting (March 22–23, 1946). At that time, representatives from the Mathematics Section of the Louisiana Teachers Association and the Louisiana Parent-Teachers Association were added to the Committee. Now there are three members upon the Committee from each of the five groups.

The most outstanding work accomplished by the Committee in 1941–42 was concerned with the secondary field and teacher training. The Committee had many sessions with the Superintendent of Education of Louisiana, his department heads, and several of his committees. These deliberative sessions were scheduled after the Joint Committee had made an extensive study of the problems at hand. The bibliography of the Committee consists of approximately twenty studies devoted to problems concerning secondary mathematics.

Before the Joint Committee was instituted, the following conditions existed in Louisiana:

1. Students could graduate from high school with only one unit of mathematics; this unit could be in algebra, general mathematics, or arithmetic.

2. Teachers needed only six semester hours of college mathematics in order to teach high school mathematics.

In 1942, the following changes were made by the State Department of Education:

1. The number of units in mathematics required for graduation from high school was increased to three. (The Committee had asked for two.) War conditions had an influence here. It is understood that this requirement will be reduced to two units.

2. The number of units of college mathematics required for certification to teach mathematics in high school was increased to eighteen. The Committee had asked for 24 hours, including 9 hours in college algebra, trigonometry, and analytic geometry (or general mathematics), 6 hours in differential and integral

calculus, 3 hours in the history of mathematics, 3 hours in college geometry, and 3 hours in the theory of equations or differential equations.

3. All teachers, irrespective of the level at which they wish to teach, must have six semester hours of college mathematics.

Because of this third requirement, the colleges in Louisiana are trying to develop a course primarily for those students who are preparing to teach in the elementary grades. Such a course would include some history of mathematics, some drill in the fundamental operations, and a survey of elementary mathematics, possibly including some calculus.

It is undoubtedly true that some of the changes made in 1942 were a result of war-time conditions and the realization of the inadequacy of mathematical training in the high schools, but it is believed that they were partially a result of the efforts of this Committee. The Committee is now trying to forestall the apparent tendency to return to the conditions that prevailed in the pre-war years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

The following eighty persons have been elected to membership on applications duly certified:

M. L. ANTHONY. U. S. Navy. 9923 Rutland St., Detroit, Mich.

J. G. BALLINGER, B.S.(Notre Dame; Illinois) Teacher, High School, Chicago, Ill.

A. F. BARTHOLOMAY, A.M.(Syracuse) Asst. Prof., Keuka Coll., Keuka Park, N. Y.

P. T. BATEMAN, A.M.(Pennsylvania) Lecturer, Statistics, Bryn Mawr Coll.; Grad. Student, Univ. of Pennsylvania, Philadelphia, Pa.

R. H. BING, Ph.D.(Texas) Asst. Prof., Univ. of Texas, Austin, Tex.

G. F. BRADFIELD, M.S.(Northwestern Univ.) Instr., Illinois Inst. of Tech.; Northwestern Univ.

L. P. BURTON, A.M.(Alabama) Instr., Univ. of Alabama, University, Ala.

M. I. CHERNOFSKY, M.B.A.(New York Univ.) 1419 Carroll St., Brooklyn 13, N. Y.

HELEN E. CLARKSON, A.M.(Duke) Adjunct Prof., Univ. of South Carolina, Columbia, S. C.

H. J. COHEN, A.B.(C.C.N.Y.) Grad. Asst., Univ. of Wisconsin, Madison 6, Wis.

B. H. COLVIN, Ph.D.(Wisconsin) Instr., Univ. of Wisconsin, Madison 6, Wis.

I. D. COOK, B.S.(Univ. of Maryland) Salesman, Automotive Parts, System Brakes, Baltimore, Md.

BYRON COSBY, JR., Ph.D.(Chicago) Instr., U. S. Naval Acad., Annapolis, Md.

SISTER AGNES THERESE DIMOND, A.M.(Columbia) Asso. Prof., Coll. of Mount St. Joseph, Mount St. Joseph, Ohio

NELLE C. DOUGLAS, M.Ed.(Duke) Instr., Univ. of South Carolina, Columbia, S. C.

H. P. EDMUNDSON. 1st Lt., Sig. Co. 539 N. Kilkea Drive, Los Angeles 36, Calif.

M. D. EULENBERG, A.M.(Loyola Univ.) Instr., Evening School, Illinois Inst. of Tech., Chicago, Ill.

H. E. FETTIS, A.B.(Wittenberg) Aero. Engr., Air Tech. Service Command. 931 Five Oaks Ave., Dayton 6, Ohio

REV. F. J. FISCHER, M.S.(DePaul) Instr., DePaul Univ., Chicago, Ill.

HARLEY FLANDERS. Student, Univ. of Chicago, Chicago, Ill.

ATHANASIUS FLORIS, Grad. Engr.(Tech. Hochschule, Munich) Asst. to Structural Engr., Pacific Elec. Ry. Co. 748 S. Kingsley Drive, Los Angeles 5, Calif.

M. K. FORT, JR., A.M.(Virginia) Instr., Univ. of Virginia, Charlottesville, Va.

MARY L. FOSTER, M.S.(Louisiana State) Adjunct Prof., Univ. of South Carolina, Columbia, S. C.

A. M. FREEMAN. Dir., Math. Lab., Boston Fiduciary and Research Associates, Boston, Mass.

SADIE C. GASAWAY, M.S.(Illinois) Instr., Tennessee A. and I. State Coll., Nashville, Tenn.

M. J. GOTTLIEB, Ph.D.(Washington Univ.) Asst. Prof., Washington Univ., St. Louis, Mo.

G. W. GROTTS, M.S.(Illinois) Instr., Univ. of Tennessee, Knoxville, Tenn.

W. S. GUSTIN, A.B.(U.C.L.A.) Asst., Univ. of California at Los Angeles, Los Angeles 24, Calif.

R. V. GUTHRIE, JR., M.S.(Iowa) Prof., Southwestern Louisiana Inst., Lafayette, La.

ISRAEL HALPERIN, Ph.D.(Princeton) Asso. Prof., Queen's Univ., Kingston, Ont., Canada

KATHARINE E. HAZARD, Ph.D.(Chicago) Instr., New Jersey Coll. for Women, Rutgers Univ., New Brunswick, N. J.

E. W. HAZLETON, A.M.(Chicago) Teacher, Calumet High School, Chicago, Ill.

REV. J. J. HENNESSEY, A.M.(Woodstock Coll.) Prof., Woodstock Coll., Woodstock, Md.

R. T. HOOD. Student, Oberlin Coll., Oberlin, Ohio

N. C. HUNSAKER, A.M.(California) Asst. Prof., Utah State Agric. Coll., Logan, Utah

H. D. HUSKEY, Ph.D.(Ohio State) Instr., Univ. of Pennsylvania, Philadelphia, Pa.

L. A. JEHN, B.M.E.(Dayton) Instr., Univ. of Dayton, Dayton, Ohio

C. A. JOHNSON, A.M.(Northwestern Univ.) Instr., Univ. of Missouri, School of Mines and Met., Rolla, Mo.

W. B. KEHL, A.M.(Harvard) Instr., Georgia School of Tech., Atlanta, Ga.

P. J. KELLY, Ph.D.(Wisconsin) Univ. of Southern California, Los Angeles, Calif.

E. R. KIELY, A.M.(Fordham) Prof., Iona Coll., New Rochelle, N. Y.

V. L. KLEE, JR., A.B.(Pomona) Grad. Student, Univ. of Virginia, Charlottesville, Va.

SISTER M. PACHOMIA LACKAY, A.M.(Missouri) Coll. of St. Teresa, Kansas City, Mo.

E. PAULINE LACY, Ed.M.(Harvard) Prof., William Woods Coll., Fulton, Mo.

L. C. LEITHOLD, JR., A.B.(California) Teaching Asst., Univ. of California, Berkeley 4, Calif.

C. C. LIN, Ph.D.(Calif. Inst. of Tech.) Asst. Prof., Brown Univ., Providence, R.I.

D. E. MARRS, A.M.(California) Teacher, Senior High School, Pittsburg, Calif.

SISTER MARY AMATOR, A.M.(Pennsylvania) Asst. Prof., Immaculata Coll., Immaculata, Pa.

E. L. MCDOWELL. Student, Illinois Inst. of Tech., Chicago, Ill.

ELSA H. MUUSS, M.S. in Educ.(Northwestern), A.M.(Loyola Univ.) Teacher, Chicago City Coll., Wright Branch, Chicago, Ill.

S. B. MYERS, Ph.D.(Harvard) Asso. Prof., Univ. of Michigan, Ann Arbor, Mich.

H. M. NAHIKIAN, Ph.D.(North Carolina) Lt. Comdr., U.S.N.R. Instr., U. S. Naval Acad., Annapolis, Md.

A. B. NEALE, B.S. in M.E.(Detroit Inst. of Tech.) Research and Develop. Engr., Watson Labs. P.O. Box 889, Red Bank, N. J.

W. J. NEMEREVER, A.M.(Iowa State) Aerodynamicist, Curtiss-Wright Corp., Airplane Div., Columbus, Ohio

J. J. NEWMAN. Student, Harvard Univ., Cambridge, Mass.

BART PARK, M.S.(Mich. Coll. of Mining and Tech.) Asso. Prof., Michigan Coll. of Mining and Tech., Houghton, Mich.

R. S. PATE, Ph.D.(Illinois) Asso. Prof., Univ. of South Carolina, Columbia, S. C.

R. P. PETERSON, JR. Student, Univ. of California at Los Angeles, Los Angeles, Calif.

PASQUALE PORCELLI. Student, Illinois Inst. of Tech., Chicago, Ill.

V. J. RATHBUN, B.S.(Univ. of Dayton) Instr., Colegio San Jose, Rio Piedras, Puerto Rico

O. W. RECHARD, A.B.(Wyoming) Research Asst., Univ. of Wisconsin, Madison 6, Wis.

DOROTHY L. REES, A.M.(Texas) Instr., Trinity Univ., San Antonio 1, Tex.

SISTER MARY EDMUND RICE, Ed.M.(Cincinnati) Business Mgr., Instr., Our Lady of Cincinnati Coll., Edgecliff, Walnut Hills, Cincinnati 6, Ohio

R. A. ROSENBAUM, A.B.(Yale) Asst. Prof., Reed Coll., Portland 2, Ore.

MARIAN C. ROSENBECK, M.S.(DePaul) Instr., Chicago City Coll., Wilson Branch, Chicago, Ill.

LEILA R. RUBASHKIN. Student, Hunter Coll., New York, N. Y.

R. G. SCHAWALDER, B.S.(Univ. of Dayton) Head of Dept., Mount St. John Prep. School, Dayton, Ohio

O. F. G. SCHILLING, Ph.D.(Marburg, Germany) Asso. Prof., Univ. of Chicago, Chicago 37, Ill.

ABRAHAM SEIDENBERG, Ph.D.(Johns Hopkins) Instr., Univ. of California, Berkeley 4, Calif.

AUDREY L. SELITZKY. Student, Univ. of Chicago, Chicago, Ill.

H. G. S. SHARMA. Prof., Lingaraj Coll., Belgaum, India

ERNST SNAPPER, Ph.D.(Princeton), Asst. Prof., Univ. of Southern California, Los Angeles 7, Calif.

W. H. SPRAGENS, JR., M.S.(Kentucky) Asso. Prof., Georgia State Woman's Coll., Valdosta, Ga.

O. M. THOMAS, A.M.(Colo. St. Coll. of Educ.) Instr., U. S. Naval Acad., Annapolis, Md.

F. B. THOMPSON. Student, Univ. of California at Los Angeles, Los Angeles, Calif.

H. J. VANDORT, A.B.(West. Mich. Coll. of Educ.) Principal, High School, LeRoy, Mich.

C. J. WEBER. Trust Officer, Chase National Bank, New York, N. Y.

C. P. WELLS, Ph.D.(Iowa State) Asst. Prof., Michigan State Coll., East Lansing, Mich.

G. S. WHITEHEAD, A.M.(Georgia; Oxford Univ.) Bay Pines, Fla.

H. J. ZIMMERBERG, Ph.D.(Chicago) Instr., North Carolina State Coll., Raleigh, N. C.

W. B. CARVER, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Twenty-eighth Summer Meeting, Ithaca, New York, August 19–20, 1946.
Thirtieth Annual Meeting, Swarthmore, Pa., December 26–27, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., October, 1946
ILLINOIS, Peoria, May 9–10, 1947
INDIANA, Terre Haute, October 18, 1946
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

WILLIAM L. HART'S

MATHEMATICS OF INVESTMENT

Third Edition

In the Third Edition • **Strengthened emphasis on the simple case for annuities certain and corresponding applications** • **Separate presentation of the general case of the theory of annuities certain after complete applications of the simple case have been met** • **Increased number of interest rates with emphasis on small rates now current in the business world.**

312 p. $2.75. With tables, 440 p. $3.60. Tables separately, $1.40.

Mathematics of Investment, Third Edition, **with Tables, bound with William L. Hart's** ***Essentials of College Algebra,*** **will be ready for fall classes.**

D. C. HEATH AND COMPANY

Boston New York Chicago Atlanta San Francisco Dallas London

TEXTBOOK NEWS

Raymond W. Brink's

ANALYTIC GEOMETRY, Revised Edition

THIS text provides a rich, complete, and adaptable course in analytic geometry with an adequate introduction to solid analytic geometry. The emphasis throughout is upon method and logic making the text especially useful as preparation for calculus and other mathematical studies. Problems, distinctive for their variety and freshness, and carefully graded as to difficulty, progress from the purely formal requiring technical skill to those demanding originality and the power of analysis. *8vo, 350 pages.* $2.90

ESSENTIALS OF ANALYTIC GEOMETRY

THIS is a simpler, more flexible, and somewhat less detailed presentation of the material in *Analytic Geometry, Revised Edition.* It provides all of the essentials in method, information, and problems for preparation for calculus and for the development of general mathematical maturity and background. *Small 8vo, 233 pages.* $2.40

D. APPLETON-CENTURY COMPANY

35 West 32nd Street New York 1, New York

MATHEMATICS OF FINANCE

by John A. Northcott

Associate Professor of Mathematics
Columbia University

This new text on the mathematics of finance develops the subject of compound interest and its application by the use of only three formulas: the Compound Interest Law, the formula for the sum of a geometric progression, and the correlative formula joining a_n and s_n. These three fundamental procedures are employed effectively for the solution of problems in this field. The author has included a chapter on logarithms and a chapter on arithmetical and geometric progression. More than 500 carefully graded problems are included. Answers are given to the odd-numbered problems.

To be published in Fall, 1946

Probably 288 pages **Probable price $3.50**

PLANE TRIGONOMETRY

by William K. Morrill

Associate Professor of Mathematics
Johns Hopkins University

The revision of this book provides the student with a complete text on the fundamentals of trigonometry. Beginning with the general definitions of the trigonometric functions, it gives the line segment representation, and introduces a simpler method for finding the functions of any angle. An entire chapter is devoted to the right triangle, and the problems in elementary physics, engineering, and navigation which appear in this chapter illustrate to the student the application of trigonometry to these sciences. Sections have been included on the use of the significant figures and the slide rule. The chapter on graphs has been enlarged, and graphs of the inverse functions, the exponential function, and the logarithmic function have been included. The text has been organized so that it is easily adaptable for use in the brief as well as the more extensive course.

To be published in Fall, 1946

Probably 256 pages **Probable price $2.50**

Rinehart & Company, Inc.
232 MADISON AVENUE · NEW YORK 16, N. Y.

COLLEGE TEXTS

GENERAL MATHEMATICS

By Currier, Watson & Frame. "The materials of the various subjects are well integrated. The explanations are concise and clear, and there are many worked examples. There is an abundance of problem material and there are many historical notes. . . . Abundant material for a year of mathematics."—*School Science and Mathematics.* 2nd Ed., $3.30.

COLLEGE ALGEBRA

By Paul R. Rider. "The book is carefully written and each chapter contains a liberal supply of graded problems. The chapter on theory of equations consists of rather more material than is to be found in many of the books of similar scope. Teachers interested in statistics will welcome the appearance of a chapter on finite differences."—*National Mathematics Magazine.* $2.20.

PLANE & SPHERICAL TRIGONOMETRY

By Paul R. Rider. "The book is written in flexible units so that it can be adapted to different types of courses. But more important than this is the clear and careful exposition, well selected illustrative examples, and excellent lists of exercises."—*American Mathematical Monthly.* With tables, $2.75. Without tables, $2.20.

DIFFERENTIAL & INTEGRAL CALCULUS

By Clyde E. Love. "One of the most widely used calculus texts."—*School Science and Mathematics.* "A very usable text. . . . The material is carefully selected and organized. The entire presentation keeps the student well in mind."—*Peabody Journal of Education.* 4th Ed., $3.60.

ANALYTIC GEOMETRY

By Clyde E. Love. "The third edition of this successful text retains the good features of the former editions. In addition the subject matter has been reorganized so as to treat the conic sections more effectively than in previous editions and at the same time require much less space. . . . The introduction to Solid Analytic Geometry is one of the best to be found in any elementary text. The exposition throughout is brief and to the point—a characteristic of all of Love's texts."—*School Science and Mathematics.* 3rd Ed., $3.00.

THE MACMILLAN COMPANY
60 FIFTH AVENUE NEW YORK 11, N. Y.

ANNEXE

Four Outstanding McGraw-Hill Books

COLLEGE ALGEBRA

By A. ADRIAN ALBERT, The University of Chicago. Ready in June

A new and rigorous approach to the subject, stressing the fundamental unity in algebra by knitting together the study of the number systems of elementary mathematics, polynomials and allied functions, algebraic identities, equations, and systems of equations. Thus algebra is presented as a unified whole, rather than as a miscellaneous collection of isolated topics.

MATHEMATICAL THEORY OF ELASTICITY

By I. S. SOKOLNIKOFF, University of Wisconsin. 373 pages, $4.50

Writing from the point of view of a mathematician, the author provides a thorough foundation in the mathematical theory of elasticity, followed by the application of the theory to problems on extension, torsion, and flexure of isotropic cylindrical bodies. There is a detailed treatment of variational methods in the theory of elasticity.

ANALYTIC GEOMETRY. New third edition

By FREDERICK S. NOWLAN, University of British Columbia. 369 pages, $2.75

As in previous editions of this well known textbook, the author's purpose has been to produce a book that is mathematically sound and at the same time easily understood and stimulating to the imagination. In the new edition the study of plane geometry is based upon the use of direction cosines, the study of conics is based upon the definition of the general conic; polar coordinates are treated from a new point of view; etc.

RUDIMENTARY MATHEMATICS FOR ECONOMISTS AND STATISTICIANS

By W. L. CRUM and JOSEPH A. SCHUMPETER, Harvard University. 203 pages, $2.50

Presents rudimentary ideas and operations essential to any effective mathematical reasoning by economists and statisticians. Covers graphic analysis, simplest case; curves and equation; limits; rates and derivatives; maxima and minima, differential equations; and determinants.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, Inc.

330 West 42nd Street **New York 18, N.Y.**

AN IMPORTANT NEW BOOK . . .
with a brand new slant on

Analytic Geometry

By F. D. MURNAGHAN, Ph. D.

A clear, highly-teachable text by an outstanding mathematician who is Chairman of the Mathematics Department at Johns Hopkins University. Professor Murnaghan's treatment of analytic geometry by the systematic use of vectors and matrices is a novel one, and greatly facilitates bridging the gap between the mathematics course, and studies of physics, engineering or calculus. Just published for use as a basic text for the freshman mathematics course, it includes 1148 problems and 20 line drawings. 368 *pages, College List,* $3.25

Calculus

By LYMAN M. KELLS, Ph. D.

"The presentation is carefully made. The illustrations are unusually clear and the figures are excellent. The problem material is varied and chosen from important fields. Finally, a superior job of printing has been done."—*Professor C. H. Richardson.* 511 *pages, over 300 illustrations, College List,* $3.75

College Algebra, with Revisions and Additional Problems

By HAROLD THAYER DAVIS

Important theories and fundamentals are introduced early in this text covering the standard algebra course in 16 chapters. Four extra chapters reveal historical incidents marking the development of algebra, biographical sketches of great mathematicians, and the meaning of mathematics—to arouse student interest. 470 *pages, College List,* $2.85

Send for your approval copies today!

PRENTICE-HALL, INC. 70 FIFTH AVENUE, NEW YORK 11

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

7

PUBLIC LIBRARY
OCT 5 - 1946
DETROIT

THE AMERICAN MATHEMATICAL MONTHLY

DEVOTED TO THE INTERESTS OF
COLLEGIATE MATHEMATICS

VOLUME 53 NUMBER 7

CONTENTS

A Simplified Approach to Cauchy's Integral Theorem . . . . . D. V. WIDDER 359
The Accuracy of Linear Interpolation . . . . P. M. HUMMEL 364
The Nodes of Some Cubic Pencils . . . . . L. E. PRIOR 366
A Manual for Young Teachers of Mathematics . H. E. BUCHANAN 371
Volume Coordinates . . . . . C. E. SPRINGER 377
Discussions and Notes . . J. L. DORROH and LYNDEN B. HOWELL 383
Clubs and Allied Activities 385
Recent Publications 388
Problems and Solutions. 394
News and Notices 401
General Information 406
The Stanford University Mathematics Examination 406
New Selective Service Policies 409
The Mathematical Association of America 411
New Members 411
Southern California Section Meeting 412
Oklahoma Section Meeting. 414
Northern California Section Meeting 416
Calendar of Future Meetings 418

AUGUST-SEPTEMBER 1946